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Optical soliton perturbation in magneto-optic waveguides

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This paper addresses the dynamics of optical solitons in the presence of perturbation terms by the aid of three integration schemes. They are modified simple equation method, trial equation scheme, and the extended trial equation scheme. There are three types of nonlinearities that are studied in this paper which are Kerr law, power law, and logarithmic law. The constraint conditions for the existence of these solitons are also presented.

Keywords: Solitons; magneto-optic waveguides; integrability; Kerr law; power law; log law.

1. Introduction

The dynamics of soliton propagation through optical fibers, photonic crystal fiber (PCF), and metamaterials is a major ongoing area of research in the field of nonlinear optics for the past few decades. There are several results that have flooded various journals with novel results.[1–15] There are a variety of results that have been reported in this area. This includes results from birefringent fibers, embedded solitons, Thirring solitons, dense wavelength-division multiplexing (DWDM) systems, cascaded systems, nematicons, and several others.[16–35] This paper will address optical solitons with a very different flavor. The effect of magnetic field will be included and the dynamics of soliton perturbation in the presence of such magnetic field will be obtained. There are three types of nonlinear media that will be studied in this paper. They are the Kerr law nonlinearity that is otherwise referred to as cubic nonlinearity, power law, and the log law. Three integration schemes are employed to obtain soliton solutions to the model. They are the modified simple equation method, trial solution method, and finally the extended trial solutions scheme. All of these methods retrieve bright solitons, dark solitons, and singular solitons while the log law nonlinearity leads to optical Gaussons. The constraint conditions for the existence of these solitons and Gaussons are also presented.

1.1. *Mathematical model*

The mathematical model that describes the dynamics of soliton propagation through optical fibers in the presence of magneto-optic field is given by the following coupled system of nonlinear Schrödinger's equation (NLSE)[5,6,10,14]:

$$\begin{aligned} &iq_t + a_1 q_{xx} + b_1 q_{xt} + \{\xi_1 F(|q|^2) + \eta_1 F(|r|^2)\}q \\ &\quad = Q_1 r + i\{\alpha_1 q_x + \beta_1 (F(|q|^2)q)_x + \nu_1 (F(|q|^2))_x q + \theta_1 F(|q|^2) q_x\}, \end{aligned} \tag{1}$$

$$\begin{aligned} &ir_t + a_2 r_{xx} + b_2 r_{xt} + \{\xi_2 F(|r|^2) + \eta_2 F(|q|^2)\}r \\ &\quad = Q_2 q + i\{\alpha_2 r_x + \beta_2 (F(|r|^2)r)_x + \nu_2 (F(|r|^2))_x r + \theta_2 F(|r|^2) r_x\}. \end{aligned} \tag{2}$$

In Eqs. (1) and (2), a_j represents the coefficients of group velocity dispersion (GVD) while b_j, for $j = 1, 2$, are the coefficients of spatio-temporal dispersion (STD). The functional F is the type of nonlinearity that will be considered. On the right-hand side, Q_j represents the magnetic field effect that avoids the formation of soliton clutter. From the perturbation terms, α_j are the coefficients of intermodal dispersion. Also, β_j represents the coefficients of self-steepening terms in order to avoid shock-wave formation, ν_j are the coefficients of nonlinear dispersion, while θ_j also gives nonlinear dispersion. On the right-hand sides, they are all treated as strong perturbation terms. This paper will carry out the integration of the model given by (1) and (2), in order to extract its soliton solutions. This will be possible, provided the type of nonlinearity is known. The subsequent three sections detail the integration schemes for three types of nonlinearity.

2. Kerr Law

For Kerr law nonlinearity, $F(s) = s$. As a consequence, Eqs. (1) and (2) modify to

$$\begin{aligned} &iq_t + a_1 q_{xx} + b_1 q_{xt} + \{\xi_1 |q|^2 + \eta_1 |r|^2\} q \\ &\quad = Q_1 r + i\{\alpha_1 q_x + \beta_1 (|q|^2 q)_x + \nu_1 (|q|^2)_x q + \theta_1 |q|^2 q_x\}, \end{aligned} \tag{3}$$

$$\begin{aligned} &ir_t + a_2 r_{xx} + b_2 r_{xt} + \{\xi_2 |r|^2 + \eta_2 |q|^2\} r \\ &\quad = Q_2 q + i\{\alpha_2 r_x + \beta_2 (|r|^2 r)_x + \nu_2 (|r|^2)_x r + \theta_2 |r|^2 r_x\}. \end{aligned} \tag{4}$$

To integrate the coupled NLSE (3)–(4), we assume a solution structure of the form

$$q(x,t) = P_1(\zeta) e^{i\phi(x,t)}, \tag{5}$$

$$r(x,t) = P_2(\zeta) e^{i\phi(x,t)}, \tag{6}$$

where the wave variable ζ is given by

$$\zeta = x - vt. \tag{7}$$

Here, $P_l(\zeta)$ for $l = 1, 2$ represents the amplitude component of the soliton and v is the speed of the soliton, while phase factor is defined as

$$\phi(x,t) = -\kappa x + \omega t + \theta, \tag{8}$$

where κ is the frequency of the solitons while ω represents the wave number and θ is the phase constant. Substituting (5) and (6) into Eqs. (3) and (4) and then decomposing into real and imaginary parts give

$$\begin{aligned} &(a_l - b_l v) P_l'' + (b_l \omega \kappa - \omega - a_l \kappa^2 - \alpha_l \kappa) P_l \\ &\quad - (\kappa(\beta_l + \theta_l) - \xi_l) P_l^3 + \eta_l P_l P_{\bar{l}}^2 - Q_l P_{\bar{l}} = 0, \end{aligned} \tag{9}$$

with $\bar{l} = 3 - l$, and imaginary parts yields

$$-v(1 - b_l \kappa) P_l' + (b_l \omega - 2 a_l \kappa - \alpha_l) P_l' = (3\beta_l + 2\nu_l + \theta_l) P_l^2 P_l'. \tag{10}$$

From Eq. (10), it is possible to retrieve the speed of the soliton

$$v = \frac{b_l \omega - 2 a_l \kappa - \alpha_l}{1 - b_l \kappa}, \tag{11}$$

as long as the constraints

$$b_l \kappa \neq 1, \tag{12}$$

$$3\beta_l + 2\nu_l + \theta_l = 0 \tag{13}$$

remain valid. Now, equating the two values of the solitons speed (11) leads to

$$a_1 = a_2, \quad b_1 = b_2, \quad \alpha_1 = \alpha_2. \tag{14}$$

Consequently, Eq. (11) reduces to

$$v = \frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}, \tag{15}$$

where it is assumed that $a_l = a$, $b_l = b$ and $\alpha_l = \alpha$ for $l = 1, 2$. Therefore, the coupled NLSE with Kerr law nonlinearity for the perturbed magneto-optic waveguide is re-casted as

$$\begin{aligned} &iq_t + aq_{xx} + bq_{xt} + \{\xi_1|q|^2 + \eta_1|r|^2\}q \\ &\quad = Q_1 r + i\{\alpha q_x + \beta_1(|q|^2 q)_x + \nu_1(|q|^2)_x q + \theta_1|q|^2 q_x\}, \end{aligned} \tag{16}$$

$$\begin{aligned} &ir_t + ar_{xx} + br_{xt} + \{\xi_2|r|^2 + \eta_2|q|^2\}r \\ &\quad = Q_2 q + i\{\alpha r_x + \beta_2(|r|^2 r)_x + \nu_2(|r|^2)_x r + \theta_2|r|^2 r_x\}, \end{aligned} \tag{17}$$

and as a consequence, the real part equation (9) modifies to

$$\begin{aligned} &(a - bv)P_l'' + (b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa)P_l + 2\kappa(\beta_l + \nu_l)P_l^3 \\ &\quad + \{\xi_l P_l^2 + \eta_l P_{\bar{l}}^2\}P_l - Q_l P_{\bar{l}} = 0, \end{aligned} \tag{18}$$

or

$$\begin{aligned} &(a - bv)P_1'' + (b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa)P_1 - (\kappa(\beta_1 + \theta_1) - \xi_1)P_1^3 \\ &\quad + \eta_1 P_1 P_2^2 - Q_1 P_2 = 0, \end{aligned} \tag{19}$$

$$\begin{aligned} &(a - bv)P_2'' + (b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa)P_2 - (\kappa(\beta_2 + \theta_2) - \xi_2)P_2^3 \\ &\quad + \eta_2 P_2 P_1^2 - Q_2 P_1 = 0. \end{aligned} \tag{20}$$

2.1. *Modified simple equation method*

In this subsection, the modified simple equation method[2,3,8] will be explored in detail to obtain bright, dark, and singular soliton solutions to Eqs. (3) and (4). Using the balancing principle in Eq. (18) leads to

$$P_{\bar{l}} = P_l. \tag{21}$$

Consequently, Eq. (18) reduces to

$$(a - bv)P_l'' + (b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa - Q_l)P_l + (\eta_l + \xi_l + 2\kappa(\beta_l + \nu_l))P_l^3 = 0. \tag{22}$$

We suppose that Eq. (22) has the formal solution as

$$P(\zeta) = \sum_{l=0}^{N} a_l \left(\frac{\psi'(\zeta)}{\psi(\zeta)}\right)^l, \tag{23}$$

where a_l are constants to be determined, such that $a_N \neq 0$, and $\psi(\zeta)$ is an unknown function to be determined later. Balancing P_l'' with P_l^3 in Eq. (22), then we get $N = 1$. So, we reach

$$P_l(\zeta) = a_0 + a_1\left(\frac{\psi'(\zeta)}{\psi(\zeta)}\right), \quad a_1 \neq 0. \tag{24}$$

Substituting Eq. (24) in Eq. (22) and then setting the coefficients of $\psi^{-j}(\zeta)$, $j = 0, 1, 2, 3$, to zero, then we obtain a set of algebraic equations involving a_0, a_1, κ, α, η_l,

ξ_l, Q_l, v and ω as follows:

ψ^{-3} coeff.:

$$a_1\big(2(a-bv)+a_1^2(\eta_l+2\kappa\beta_l+2\kappa\nu_l+\xi_l)\big)(\psi')^3=0, \tag{25}$$

ψ^{-2} coeff.:

$$3a_1(\psi''(bv-a)+a_0a_1\psi'(\eta_l+2\kappa\beta_l+2\kappa\nu_l+\xi_l))\psi'=0, \tag{26}$$

ψ^{-1} coeff.:

$$-a_1(\psi^{(3)}(bv-a)+\psi'(a\kappa^2+\alpha\kappa-3a_0^2(\eta_l+2\kappa\beta_l+2\kappa\nu_l+\xi_l)-b\kappa\omega+Q_l+\omega))=0, \tag{27}$$

ψ^0 coeff.:

$$a_0(-a\kappa^2-\alpha\kappa+a_0^2(\eta_l+2\kappa\beta_l+2\kappa\nu_l+\xi_l)+b\kappa\omega-Q_l-\omega)=0. \tag{28}$$

Solving this system, we obtain

$$a_0=\pm\sqrt{\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_l+\omega}{\eta_l+2\kappa\beta_l+2\kappa\nu_l+\xi_l}},\quad a_1=\mp\sqrt{-\frac{2(a-bv)}{\eta_l+2\kappa\beta_l+2\kappa\nu_l+\xi_l}}, \tag{29}$$

and

$$\psi''=\sqrt{-\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_l+\omega)}{a-bv}}\,\psi', \tag{30}$$

$$\psi'''=-\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_l+\omega)}{a-bv}\,\psi'. \tag{31}$$

From Eqs. (30) and (31), we can deduce that

$$\psi'(\zeta)=\sqrt{-\frac{a-bv}{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_l+\omega)}}c_1e^{\sqrt{-\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_l+\omega)}{a-bv}}\zeta}, \tag{32}$$

$$\psi(\zeta)=-\frac{a-bv}{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_l+\omega)}c_1e^{\sqrt{-\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_l+\omega)}{a-bv}}\zeta}+c_2, \tag{33}$$

where c_1 and c_2 are constants of integration. Substituting Eqs. (32) and (33) into Eq. (24), we obtain the following exact solution to Eqs. (3) and (4):

$$q(x,t)=\pm\left\{\sqrt{\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega}{\eta_1+\xi_1+2\kappa(\beta_1+\nu_1)}}+\frac{\frac{a-bv}{\sqrt{(\eta_1+\xi_1+2\kappa(\beta_1+\nu_1))(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega)}}\times c_1e^{\sqrt{-\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega)}{a-bv}}\zeta}}{-\frac{a-bv}{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega)}\times c_1e^{\sqrt{-\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega)}{a-bv}}\zeta}+c_2}\right\}\times e^{i(-\kappa x+\omega t+\theta)}, \tag{34}$$

$$r(x,t) = \pm\left\{\sqrt{\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega}{\eta_2+\xi_2+2\kappa(\beta_2+\nu_2)}} + \frac{\frac{a-bv}{\sqrt{(\eta_2+\xi_2+2\kappa(\beta_2+\nu_2))(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)}} \times c_1 e^{\sqrt{-\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)}{a-bv}}\zeta}}{-\frac{a-bv}{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)} \times c_1 e^{\sqrt{-\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)}{a-bv}}\zeta} + c_2}\right\} \times e^{i(-\kappa x+\omega t+\theta)}. \quad (35)$$

If we set $c_1 = -\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)}{a-bv} e^{\sqrt{-\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)}{a-bv}}\zeta_0}$, $c_2 = \pm 1$, we obtain

$$q(x,t) = \pm\sqrt{\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega}{\eta_1+\xi_1+2\kappa(\beta_1+\nu_1)}} \times \tanh\left[\sqrt{-\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega}{2(a-bv)}}(x-vt+\zeta_0)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (36)$$

$$r(x,t) = \pm\sqrt{\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega}{\eta_2+\xi_2+2\kappa(\beta_2+\nu_2)}} \times \tanh\left[\sqrt{-\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega}{2(a-bv)}}(x-vt+\zeta_0)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (37)$$

or

$$q(x,t) = \pm\sqrt{\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega}{\eta_1+\xi_1+2\kappa(\beta_1+\nu_1)}} \times \coth\left[\sqrt{-\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega}{2(a-bv)}}(x-vt+\zeta_0)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (38)$$

$$r(x,t) = \pm\sqrt{\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega}{\eta_2+\xi_2+2\kappa(\beta_2+\nu_2)}} \times \coth\left[\sqrt{-\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega}{2(a-bv)}}(x-vt+\zeta_0)\right]e^{i(-\kappa x+\omega t+\theta)}. \quad (39)$$

These are dark and singular soliton solutions, respectively, and are valid for

$$a\kappa^2+\alpha\kappa-b\kappa\omega+Q_l+\omega > 0, \quad (40)$$

$$a-bv < 0. \quad (41)$$

2.2. *Trial equation method*

In this subsection, we would like to extend the trial equation method[1,4,13] to solve Eqs. (3) and (4). To kick off the solution extraction process to (22), the following trial equation is considered:

$$(P')^2 = F(P) = \sum_{l=0}^{s} a_l P^l, \tag{42}$$

where s and a_l are constants to be determined. We rewrite Eq. (42) by the integral form

$$\pm(\zeta - \zeta_0) = \int \frac{dP}{\sqrt{F(P)}}. \tag{43}$$

Balancing P_l'' with P_l^3 in Eq. (22), then we get $s = 4$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

u^3 coeff.:

$$2a_4(a - bv) + \eta_l + 2\kappa\beta_l + 2\kappa\nu_l + \xi_l = 0, \tag{44}$$

u^2 coeff.:

$$\frac{3}{2}a_3(a - bv) = 0, \tag{45}$$

u^1 coeff.:

$$a_2(a - bv) - a\kappa^2 - \alpha\kappa + b\kappa\omega - Q_l - \omega = 0, \tag{46}$$

u^0 coeff.:

$$\frac{1}{2}a_1(a - bv) = 0. \tag{47}$$

Solving the above system of algebraic equations, we obtain the following results:

$$\begin{aligned} &a_1 = 0, \quad a_2 = \frac{a\kappa^2 + \alpha\kappa - b\kappa\omega + Q_l + \omega}{a - bv}, \quad a_3 = 0, \\ &a_4 = -\frac{\eta_l + \xi_l + 2\kappa(\beta_l + \nu_l)}{2(a - bv)}. \end{aligned} \tag{48}$$

Substituting these results into Eqs. (42) and (43), we get

$$\pm(\zeta - \zeta_0) = \int \frac{dP_l}{\sqrt{a_0 + \frac{a\kappa^2 + \alpha\kappa - b\kappa\omega + Q_l + \omega}{a - bv} P_l^2 - \frac{\eta_l + \xi_l + 2\kappa(\beta_l + \nu_l)}{2(a - bv)} P_l^4}}, \tag{49}$$

where a_0 is an arbitrary real constant. If we set $a_0 = 0$ in Eq. (49) and integrating with respect to P_l, we get

$$\begin{aligned} q(x,t) = \pm &\sqrt{-\frac{2(a\kappa^2 + \alpha\kappa - b\kappa\omega + Q_1 + \omega)}{\eta_1 + 2\kappa(\beta_1 + \nu_1) + \xi_1}} \\ &\times \operatorname{sech}\left[\sqrt{\frac{a\kappa^2 + \alpha\kappa - b\kappa\omega + Q_1 + \omega}{a - bv}}(x - vt - \zeta_0)\right] e^{i(-\kappa x + \omega t + \theta)}, \end{aligned} \tag{50}$$

$$r(x,t) = \pm\sqrt{-\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)}{\eta_2+2\kappa(\beta_2+\nu_2)+\xi_2}} \times \operatorname{sech}\left[\sqrt{\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega}{a-bv}}(x-vt-\zeta_0)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (51)$$

or

$$q(x,t) = \pm\sqrt{\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega)}{\eta_1+2\kappa(\beta_1+\nu_1)+\xi_1}} \times \operatorname{csch}\left[\sqrt{\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega}{a-bv}}(x-vt-\zeta_0)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (52)$$

$$r(x,t) = \pm\sqrt{\frac{2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)}{\eta_2+2\kappa(\beta_2+\nu_2)+\xi_2}} \times \operatorname{csch}\left[\sqrt{\frac{a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega}{a-bv}}(x-vt-\zeta_0)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (53)$$

which are bright and singular soliton solutions and they exist for

$$a\kappa^2+\alpha\kappa-b\kappa\omega+Q_l+\omega > 0, \quad (54)$$

$$a-bv > 0. \quad (55)$$

2.3. *Extended trial equation method*

In this subsection, we will apply the extended trial equation method[7,9,11,12,15] to address the coupled NLSE with Kerr law nonlinearity. To start with the extraction of solutions to (19) and (20), the following assumption for the soliton structure is made:

$$P_1 = \sum_{i=0}^{\varsigma} \tau_i \Psi^i, \quad (56)$$

$$P_2 = \sum_{i=0}^{\tilde{\varsigma}} \tilde{\tau}_i \Psi^i, \quad (57)$$

where

$$(\Psi')^2 = \Lambda(\Psi) = \frac{\Phi(\Psi)}{\Upsilon(\Psi)} = \frac{\mu_\sigma\Psi^\sigma+\cdots+\mu_1\Psi+\mu_0}{\chi_\rho\Psi^\rho+\cdots+\chi_1\Psi+\chi_0}. \quad (58)$$

Here $\tau_0, \dots, \tau_\varsigma$; $\tilde{\tau}_0, \dots, \tilde{\tau}_{\tilde{\varsigma}}$; $\mu_0, \dots, \mu_\sigma$ and $\chi_0, \dots, \chi_\rho$ are constants to be determined later. We can reduce Eq. (58) to the elementary integral form as follows:

$$\pm(\zeta - \zeta_0) = \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}} = \int \sqrt{\frac{\Upsilon(\Psi)}{\Phi(\Psi)}} d\Psi. \tag{59}$$

According to the balance principle, one determines a relation of σ, ρ, ς and $\tilde{\varsigma}$ which is given by

$$\varsigma = \tilde{\varsigma} = \frac{\sigma - \rho - 2}{2}. \tag{60}$$

When $\sigma = 4$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (60), we then assume that Eqs. (19) and (20) have the following formal solutions:

$$P_1 = \tau_0 + \tau_1 \Psi, \tag{61}$$

$$P_2 = \tilde{\tau}_0 + \tilde{\tau}_1 \Psi, \tag{62}$$

where τ_i and $\tilde{\tau}_i$ for $i = 0, 1$ are constants to be determined later, and Ψ satisfies Eq. (58). Substituting these formal solutions into (19) and (20), and solving the resulting system of algebraic equations, one recovers

$$\begin{aligned}
\mu_1 &= \frac{2\tilde{\tau}_0[\mu_2 \tilde{\tau}_1^2(a - bv) + 2\chi_0 \tilde{\tau}_0^2(\tilde{\tau}_1^2(\xi_2 - \kappa(\theta_2 + \beta_2)) + \eta_2 \tau_1^2)]}{\tilde{\tau}_1^3(a - bv)}, \\
\mu_3 &= \frac{2\chi_0 \tilde{\tau}_0[\tilde{\tau}_1^2(\kappa(\theta_2 + \beta_2) - \xi_2) - \eta_2 \tau_1^2]}{\tilde{\tau}_1(a - bv)}, \\
\mu_4 &= \frac{\chi_0[\tilde{\tau}_1^2(\kappa(\theta_2 + \beta_2) - \xi_2) - \eta_2 \tau_1^2]}{2(a - bv)}, \\
\xi_1 &= \eta_2 + \kappa(\theta_1 + \beta_1) - \frac{\tilde{\tau}_1^2[\eta_1 + \kappa(\theta_2 + \beta_2) - \xi_2]}{\tau_1^2}, \\
\mu_0 &= \mu_0, \quad \mu_2 = \mu_2, \quad \tau_1 = \tau_1, \quad \tilde{\tau}_0 = \tilde{\tau}_0, \quad \tilde{\tau}_1 = \tilde{\tau}_1, \quad \tau_0 = \frac{\tau_1 \tilde{\tau}_0}{\tilde{\tau}_1}, \quad Q_1 = \frac{\tau_1^2 Q_2}{\tilde{\tau}_1^2}, \\
\omega &= \frac{-3\eta_2 \tau_1^2 \chi_0 \tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1 \chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a) + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2 + \beta_2) - \xi_2)))]}{\chi_0 \tilde{\tau}_1^2(b\kappa - 1)}.
\end{aligned} \tag{63}$$

Substituting these results into (58) and (59) leads to

$$\pm(\zeta - \zeta_0) = \Omega \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{64}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4}, \quad \Omega = \sqrt{\frac{\chi_0}{\mu_4}}. \tag{65}$$

As a consequence, we obtain the traveling wave solutions to the coupled NLSE with Kerr law nonlinearity in the following forms:

For $\Lambda(\Psi) = (\Psi - \lambda_1)^4$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 \pm \frac{\tau_1\Omega}{x - \{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\}t - \zeta_0}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a)\\ + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2 + \beta_2) - \xi_2)))]\end{array}}{\chi_0\tilde{\tau}_1^2(b\kappa - 1)}\right)t + \theta\right\}\right]. \tag{66}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 \pm \frac{\tilde{\tau}_1\Omega}{x - \{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\}t - \zeta_0}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a)\\ + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2 + \beta_2) - \xi_2)))]\end{array}}{\chi_0\tilde{\tau}_1^2(b\kappa - 1)}\right)t + \theta\right\}\right]. \tag{67}$$

If $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{4\tau_1\Omega^2(\lambda_2 - \lambda_1)}{4\Omega^2 - \left[(\lambda_1 - \lambda_2)\left(x - \{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\}t - \zeta_0\right)\right]^2}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a)\\ + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2 + \beta_2) - \xi_2)))]\end{array}}{\chi_0\tilde{\tau}_1^2(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{68}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \frac{4\tilde{\tau}_1\Omega^2(\lambda_2 - \lambda_1)}{4\Omega^2 - \left[(\lambda_1 - \lambda_2)\left(x - \{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\}t - \zeta_0\right)\right]^2}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a)\\ + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2 + \beta_2) - \xi_2)))]\end{array}}{\chi_0\tilde{\tau}_1^2(b\kappa - 1)}\right)t + \theta\right\}\right]. \tag{69}$$

However, when $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_2 + \frac{\tau_1(\lambda_2 - \lambda_1)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega}\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t - \zeta_0\right)\right] - 1}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)}\right)t + \theta\right\}\right], \tag{70}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_2 + \frac{\tilde{\tau}_1(\lambda_2 - \lambda_1)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega}\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t - \zeta_0\right)\right] - 1}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)}\right)t + \theta\right\}\right], \tag{71}$$

and

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{\tau_1(\lambda_1 - \lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega}\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t - \zeta_0\right)\right] - 1}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)}\right)t + \theta\right\}\right], \tag{72}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \frac{\tilde{\tau}_1(\lambda_1 - \lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega}\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t - \zeta_0\right)\right] - 1}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)}\right)t + \theta\right\}\right]. \tag{73}$$

Whenever $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$,

$$q(x,t) = \left\{\begin{array}{l}\tau_0 + \tau_1\lambda_1 - \dfrac{2\tau_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2)\cosh} \\ \qquad \times \left[\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{\Omega}\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1-b\kappa}\right\}t\right)\right]\end{array}\right\}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{l}-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) \\ \quad + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2) - \xi_2)))]\end{array}}{\chi_0\tilde{\tau}_1^2(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{74}$$

$$r(x,t) = \left\{\begin{array}{l}\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 - \dfrac{2\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2)\cosh} \\ \qquad \times \left[\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{\Omega}\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1-b\kappa}\right\}t\right)\right]\end{array}\right\}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{l}-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) \\ \quad + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2) - \xi_2)))]\end{array}}{\chi_0\tilde{\tau}_1^2(b\kappa - 1)}\right)t + \theta\right\}\right]. \tag{75}$$

On the other hand, if $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$,

$$q(x,t) = \left\{\begin{array}{l}\tau_0 + \tau_1\lambda_2 + \dfrac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4)\mathrm{sn}^2} \\ \qquad \times \left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega}\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1-b\kappa}\right\}t - \zeta_0\right), m\right]\end{array}\right\}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{l}-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) \\ \quad + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2) - \xi_2)))]\end{array}}{\chi_0\tilde{\tau}_1^2(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{76}$$

$$r(x,t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1\lambda_2 + \frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\mathrm{sn}^2 \times \left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t-\zeta_0\right),m\right]} \right\}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2+\tilde{\tau}_1[\tau_1\chi_0Q_2+\tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa)+3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)}\right)t+\theta\right\}\right], \tag{77}$$

where modulus m is given by

$$m^2 = \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \tag{78}$$

It should be noted that λ_j for $j = 1, \ldots, 4$ are the roots of the following equation:

$$\Lambda(\Psi) = 0. \tag{79}$$

Under the conditions $\tau_0 = -\tau_1\lambda_1$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_1$ and $\zeta_0 = 0$, the solutions (66)–(75) can be reduced to plane wave solutions:

$$q(x,t) = \left\{\pm\frac{\tau_1\Omega}{x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t}\right\}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2+\tilde{\tau}_1[\tau_1\chi_0Q_2+\tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa)+3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)}\right)t+\theta\right\}\right], \tag{80}$$

$$r(x,t) = \left\{\pm\frac{\tilde{\tau}_1\Omega}{x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t}\right\}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2+\tilde{\tau}_1[\tau_1\chi_0Q_2+\tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa)+3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)}\right)t+\theta\right\}\right], \tag{81}$$

$$q(x,t) = \left\{ \frac{4\tau_1 \Omega^2(\lambda_2-\lambda_1)}{4\Omega^2 - \left[(\lambda_1-\lambda_2)\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]^2} \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)} \right) t + \theta \right\} \right], \tag{82}$$

$$r(x,t) = \left\{ \frac{4\tilde{\tau}_1 \Omega^2(\lambda_2-\lambda_1)}{4\Omega^2 - \left[(\lambda_1-\lambda_2)\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]^2} \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)} \right) t + \theta \right\} \right], \tag{83}$$

traveling wave solutions

$$q(x,t) = \left\{ \frac{\tau_1(\lambda_2-\lambda_1)}{2}\left(1 \mp \coth\left[\frac{\lambda_1-\lambda_2}{2\Omega}\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]\right) \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)} \right) t + \theta \right\} \right], \tag{84}$$

$$r(x,t) = \left\{ \frac{\tilde{\tau}_1(\lambda_2-\lambda_1)}{2}\left(1 \mp \coth\left[\frac{\lambda_1-\lambda_2}{2\Omega}\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]\right) \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)} \right) t + \theta \right\} \right], \tag{85}$$

and soliton solutions

$$q(x,t) = \left\{ \frac{K}{M + \cosh\left[L\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]} \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv-a) + \chi_0(\kappa(\alpha+a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2+\beta_2)-\xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa-1)} \right) t + \theta \right\} \right], \tag{86}$$

$$r(x,t) = \left\{\frac{\tilde{K}}{M + \cosh\left[L\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right)\right]}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a) \\ + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2 + \beta_2) - \xi_2)))]\end{array}}{\chi_0\tilde{\tau}_1^2(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{87}$$

where

$$K = \frac{2\tau_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3 - \lambda_2}, \tag{88}$$

$$\tilde{K} = \frac{2\tilde{\tau}_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3 - \lambda_2}, \tag{89}$$

$$L = \frac{\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{\Omega}, \tag{90}$$

$$M = \frac{2\lambda_1 - \lambda_2 - \lambda_3}{\lambda_3 - \lambda_2}. \tag{91}$$

Note that the amplitudes of the solitons are given by (88) and (89), while the inverse width of the solitons is given by (90). These solitons are valid for $\tau_1 < 0$ and $\tilde{\tau}_1 < 0$. Moreover, under the conditions $\tau_0 = -\tau_1\lambda_2$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_2$ and $\zeta_0 = 0$, Jacobi elliptic function solutions (76) and (77) are reduced to

$$q(x,t) = \left\{\frac{K_1}{M_1 + \mathrm{sn}^2\left[L_j\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right), \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}\right]}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a) \\ + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2 + \beta_2) - \xi_2)))]\end{array}}{\chi_0\tilde{\tau}_1^2(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{92}$$

$$r(x,t) = \left\{\frac{\tilde{K}_1}{M_1 + \mathrm{sn}^2\left[L_j\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right), \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}\right]}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a) \\ + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2 + \beta_2) - \xi_2)))]\end{array}}{\chi_0\tilde{\tau}_1^2(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{93}$$

where

$$K_1 = \frac{\tau_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_1 - \lambda_4}, \tag{94}$$

$$\tilde{K}_1 = \frac{\tilde{\tau}_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_1 - \lambda_4}, \tag{95}$$

$$M_1 = \frac{\lambda_4 - \lambda_2}{\lambda_1 - \lambda_4}, \tag{96}$$

$$L_j = \frac{(-1)^j\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2\Omega} \quad \text{for } j = 1, 2. \tag{97}$$

Remark 1. When the modulus $m \to 1$, singular optical soliton solutions are recovered:

$$q(x,t) = \left\{\frac{K_1}{M_1 + \tanh^2\left[L_j\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right)\right]}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^{\,2} + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a) + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^{\,2}(\kappa(\theta_2 + \beta_2) - \xi_2)))]}{\chi_0\tilde{\tau}_1^{\,2}(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{98}$$

$$r(x,t) = \left\{\frac{\tilde{K}_1}{M_1 + \tanh^2\left[L_j\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right)\right]}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^{\,2} + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a) + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^{\,2}(\kappa(\theta_2 + \beta_2) - \xi_2)))]}{\chi_0\tilde{\tau}_1^{\,2}(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{99}$$

where $\lambda_3 = \lambda_4$.

Remark 2. However, if $m \to 0$, the following periodic singular solutions are obtained:

$$q(x,t) = \left\{\frac{K_1}{M_1 + \sin^2\left[L_j\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right)\right]}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^{\,2} + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a) + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^{\,2}(\kappa(\theta_2 + \beta_2) - \xi_2)))]}{\chi_0\tilde{\tau}_1^{\,2}(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{100}$$

$$r(x,t) = \left\{\frac{\tilde{K}_1}{M_1 + \sin^2\left[L_j\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1-b\kappa}\right\}t\right)\right]}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{-3\eta_2\tau_1^2\chi_0\tilde{\tau}_0^2 + \tilde{\tau}_1[\tau_1\chi_0 Q_2 + \tilde{\tau}_1(\mu_2(bv - a) + \chi_0(\kappa(\alpha + a\kappa) + 3\tilde{\tau}_0^2(\kappa(\theta_2 + \beta_2) - \xi_2)))]}{\chi_0\tilde{\tau}_1^2(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{101}$$

where $\lambda_2 = \lambda_3$.

3. Power Law

For the power law nonlinearity, $F(s) = s^n$, where n represents the power law nonlinearity parameter. Here the stability issue dictates $0 < n < 2$ and also $n \neq 2$ for avoiding self-focusing singularity. Thus, the system (1)–(2) is rewritten as

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + \{\xi_1|q|^{2n} + \eta_1|r|^{2n}\}q = Q_1 r + i\{\alpha_1 q_x + \beta_1(|q|^{2n}q)_x + \nu_1(|q|^{2n})_x q + \theta_1|q|^{2n}q_x\}, \tag{102}$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + \{\xi_2|r|^{2n} + \eta_2|q|^{2n}\}r = Q_2 q + i\{\alpha_2 r_x + \beta_2(|r|^{2n}r)_x + \nu_2(|r|^{2n})_x r + \theta_2|r|^{2n}r_x\}. \tag{103}$$

Upon substituting (5) and (6) into (102) and (103), the resulting real part obtained is

$$(a_l - b_l v)P_l'' + (b_l\omega\kappa - \omega - a_l\kappa^2 - \alpha_l\kappa)P_l + (\xi_l P_l^{2n} + \eta_l P_{\bar{l}}^{2n})P_l = Q_l P_{\bar{l}} + \kappa(\beta_l + \theta_l)P_l^{2n+1}, \tag{104}$$

and for the imaginary part

$$v(b_l\kappa - 1)P_l' + (b_l\omega - 2a_l\kappa - \alpha_l)P_l' = \{(2n+1)\beta_l + 2n\nu_l + \theta_l\}P_l^{2n}P_l'. \tag{105}$$

From (105) it is possible to retrieve the solitons speed (11) as long as the constraints (12) and

$$(2n+1)\beta_l + 2n\nu_l + \theta_l = 0. \tag{106}$$

Consequently (14) and (15) are also satisfied in this case, and the real part (104) becomes

$$(a - bv)P_l'' + (b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa)P_l + 2n\kappa(\beta_l + \nu_l)P_l^{2n+1} + \{\xi_l P_l^{2n} + \eta_l P_{\bar{l}}^{2n}\}P_l - Q_l P_{\bar{l}} = 0, \tag{107}$$

or

$$(a - bv)P_l'' + (b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa)P_l + (\xi_l P_l^{2n} + \eta_l P_{\bar{l}}^{2n})P_l = Q_l P_{\bar{l}} + \kappa(\beta_l + \theta_l)P_l^{2n+1}. \tag{108}$$

3.1. *Modified simple equation method*

In this subsection, the modified simple equation approach will be utilized in detail, to seek bright, dark, and singular soliton solutions to the coupled NLSE with power law nonlinearity. Using the balancing principle in Eq. (107) leads to

$$P_{\bar{l}} = P_l. \tag{109}$$

Consequently, Eq. (107) reduces to

$$(a - bv)P_l'' + (b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa - Q_l)P_l + (\eta_l + \xi_l + 2n\kappa(\beta_l + \nu_l))P_l^{2n+1} = 0. \tag{110}$$

Set

$$P_l = U_l^{\frac{1}{n}}, \tag{111}$$

so that (110) transforms to

$$\begin{aligned}&(a - bv)(nU_lU_l'' + (1-n)U_l'^2) + n^2(b\omega\kappa - a\kappa^2 - \alpha\kappa - Q_l - \omega)U_l^2\\&\quad + n^2(\eta_l + 2\kappa n(\beta_l + \nu_l) + \xi_l)U_l^4 = 0.\end{aligned} \tag{112}$$

Balancing U_lU_l'' with U_l^4 in Eq. (112), then we get $N = 1$. Thus, we reach

$$U_l(\zeta) = a_0 + a_1\left(\frac{\psi'(\zeta)}{\psi(\zeta)}\right), \quad a_1 \neq 0. \tag{113}$$

Substituting Eq. (113) in Eq. (112) and then setting the coefficients of $\psi^{-j}(\zeta)$, $j = 0, 1, 2, 3, 4$ to zero, then we obtain a set of algebraic equations involving a_0, a_1, κ, α, η_l, ξ_l, Q_l, v and ω as follows:

ψ^{-4} coeff.:

$$a_1^2((n+1)(a - bv) + a_1^2n^2(\eta_l + 2\kappa n\beta_l + 2\kappa n\nu_l + \xi_l))\psi'^4 = 0, \tag{114}$$

ψ^{-3} coeff.:

$$\begin{aligned}&a_1(2a_0n\psi'((a - bv) + 2a_1^2n(\eta_l + 2\kappa n\beta_l + 2\kappa n\nu_l + \xi_l))\\&\quad - a_1(n+2)\psi''(a - bv))\psi'^2 = 0,\end{aligned} \tag{115}$$

ψ^{-2} coeff.:

$$\begin{aligned}&-3a_1a_0n\psi'\psi''(a - bv) + a_1^2n^2(\psi')^2(-a\kappa^2 - \alpha\kappa + 6a_0^2(\eta_l + 2\kappa n\beta_l + 2\kappa n\nu_l + \xi_l)\\&\quad + b\kappa\omega - Q_l - \omega) - a_1^2(n-1)(\psi'')^2(a - bv) + a_1^2n\psi'''\psi'(a - bv) = 0,\end{aligned} \tag{116}$$

ψ^{-1} coeff.:

$$\begin{aligned}&a_0a_1n(\psi^{(3)}(a - bv) + 2n\psi'(-a\kappa^2 - \alpha\kappa + 2a_0^2(\eta_l + 2\kappa n\beta_l + 2\kappa n\nu_l + \xi_l)\\&\quad + b\kappa\omega - Q_l - \omega)) = 0,\end{aligned} \tag{117}$$

ψ^0 coeff.:

$$a_0^2n^2(-a\kappa^2 - \alpha\kappa + a_0^2(\eta_l + 2\kappa n\beta_l + 2\kappa n\nu_l + \xi_l) + b\kappa\omega - Q_l - \omega) = 0. \tag{118}$$

Solving this system, we obtain

$$a_0 = 0, \quad a_1 = \frac{\sqrt{(n+1)(bv-a)}}{\sqrt{n^2(\eta_l + 2\kappa n\beta_l + 2\kappa n\nu_l + \xi_l)}}, \tag{119}$$

and

$$\psi'' = 0, \tag{120}$$

$$\psi''' = \frac{n(a\kappa^2 + \alpha\kappa - b\kappa\omega + Q_l + \omega)}{a - bv}\psi'. \tag{121}$$

This leads to trivial solution. Thus, modified simple equation method fails to retrieve soliton solutions in magneto-optic waveguides that carry power law nonlinearity. This is expected as it is known from before that dark and singular solitons for power law nonlinear medium will exist, provided power law collapses to Kerr law.

3.2. *Trial equation method*

This subsection will apply the trial equation approach to retrieve bright, dark, and singular soliton solutions to the governing equation. Balancing $U_l U_l''$ with U_l^4 in Eq. (112), then we get $s = 4$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

u^4 coeff.:

$$a_4(n+1)(a-bv) + n^2(\eta_l + 2\kappa n\beta_l + 2\kappa n\nu_l + \xi_l) = 0, \tag{122}$$

u^3 coeff.:

$$\frac{1}{2}a_3(n+2)(a-bv) = 0, \tag{123}$$

u^2 coeff.:

$$a_2(a-bv) - n^2(a\kappa^2 + \alpha\kappa - b\kappa\omega + Q_l + \omega) = 0, \tag{124}$$

u^1 coeff.:

$$\frac{1}{2}a_1(n-2)(a-bv) = 0, \tag{125}$$

u^0 coeff.:

$$a_0(n-1)(a-bv) = 0. \tag{126}$$

Solving the above system of algebraic equations, we obtain the following results:

$$\begin{gathered} a_0 = 0, \quad a_1 = 0, \quad a_2 = \frac{n^2(a\kappa^2 + \alpha\kappa - b\kappa\omega + Q_l + \omega)}{a - bv}, \\ a_3 = 0, \quad a_4 = -\frac{n^2(\eta_l + 2\kappa n(\beta_l + \nu_l) + \xi_l)}{(n+1)(a-bv)}. \end{gathered} \tag{127}$$

Substituting these results into Eqs. (42) and (43), we get

$$\pm(\zeta - \zeta_0) = \int \frac{dU_l}{\sqrt{\frac{n^2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_l+\omega)}{a-bv}U_l{}^2 - \frac{n^2(\eta_l+2\kappa n(\beta_l+\nu_l)+\xi_l)}{(n+1)(a-bv)}U_l{}^4}}. \tag{128}$$

Integrating with respect to U_l, we get

$$\begin{aligned} q(x,t) = {} & \pm\left\{\sqrt{-\frac{(n+1)(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega)}{\eta_l+2\kappa n(\beta_1+\nu_1)+\xi_1}}\right. \\ & \left.\times \operatorname{sech}\left[\sqrt{\frac{n^2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega)}{a-bv}}(x-vt-\zeta_0)\right]\right\}^{\frac{1}{n}} \\ & \times e^{i(-\kappa x+\omega t+\theta)}, \end{aligned} \tag{129}$$

$$\begin{aligned} r(x,t) = {} & \pm\left\{\sqrt{-\frac{(n+1)(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)}{\eta_2+2\kappa n(\beta_2+\nu_2)+\xi_2}}\operatorname{sech}\right. \\ & \left.\times\left[\sqrt{\frac{n^2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)}{a-bv}}(x-vt-\zeta_0)\right]\right\}^{\frac{1}{n}} \\ & \times e^{i(-\kappa x+\omega t+\theta)}, \end{aligned} \tag{130}$$

or

$$\begin{aligned} q(x,t) = {} & \pm\left\{\sqrt{\frac{(n+1)(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega)}{\eta_l+2\kappa n(\beta_1+\nu_1)+\xi_1}}\operatorname{csch}\right. \\ & \left.\times\left[\sqrt{\frac{n^2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_1+\omega)}{a-bv}}(x-vt-\zeta_0)\right]\right\}^{\frac{1}{n}} \\ & \times e^{i(-\kappa x+\omega t+\theta)}, \end{aligned} \tag{131}$$

$$\begin{aligned} r(x,t) = {} & \pm\left\{\sqrt{\frac{(n+1)(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)}{\eta_2+2\kappa n(\beta_2+\nu_2)+\xi_2}}\operatorname{csch}\right. \\ & \left.\times\left[\sqrt{\frac{n^2(a\kappa^2+\alpha\kappa-b\kappa\omega+Q_2+\omega)}{a-bv}}(x-vt-\zeta_0)\right]\right\}^{\frac{1}{n}} \\ & \times e^{i(-\kappa x+\omega t+\theta)}. \end{aligned} \tag{132}$$

These are again bright and singular soliton solutions and remain valid for

$$a\kappa^2+\alpha\kappa-b\kappa\omega+Q_l+\omega > 0, \tag{133}$$

$$a - bv > 0. \tag{134}$$

3.3. *Extended trial equation method*

In this subsection, we will use the extended trial equation scheme to derive soliton solutions to the model. Using the balancing procedure between P_l'' and P_l^{2n+1} in Eq. (108), we have

$$N = \frac{1}{n}. \tag{135}$$

To obtain an analytic solution, we use the transformation

$$P_1 = V_1^{\frac{1}{2n}} = V_2^{\frac{1}{2n}} = P_2 \tag{136}$$

in Eq. (108) to find

$$(a - bv)\{(1-2n)(V_1')^2 + 2nV_1V_1''\} + 4n^2(b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa - Q_1)V_1^2 + 4n^2\{\xi_1 + \eta_1 - \kappa(\beta_1 + \theta_1)\}V_1^3 = 0, \tag{137}$$

$$(a - bv)\{(1-2n)(V_2')^2 + 2nV_2V_2''\} + 4n^2(b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa - Q_2)V_2^2 + 4n^2\{\xi_2 + \eta_2 - \kappa(\beta_2 + \theta_2)\}V_2^3 = 0. \tag{138}$$

Balancing the order of V_lV_l'' and V_l^3 in Eqs. (137) and (138), we determine a relation of σ, ρ, ς and $\tilde{\varsigma}$ as

$$\varsigma = \tilde{\varsigma} = \sigma - \rho - 2. \tag{139}$$

Case 1. When $\sigma = 3$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (139), Eqs. (137) and (138) have the solutions in the forms

$$V_1 = \tau_0 + \tau_1\Psi, \tag{140}$$

$$V_2 = \tilde{\tau}_0 + \tilde{\tau}_1\Psi, \tag{141}$$

where τ_i and $\tilde{\tau}_i$ for $i = 0, 1$ are constants to be determined later, and Ψ satisfies Eq. (58). Substituting these solutions into Eqs. (137) and (138), and solving the resulting system of algebraic equations, one recovers the following solution set:

$$\mu_1 = \frac{2\mu_0\tilde{\tau}_1}{\tilde{\tau}_0} - \frac{4n^2\chi_0\tilde{\tau}_0^2[\eta_2 - \kappa(\theta_2 + \beta_2) + \xi_2]}{\tilde{\tau}_1(n+1)(a - bv)},$$

$$\mu_2 = \frac{\mu_0\tilde{\tau}_1^2}{\tilde{\tau}_0^2} - \frac{8n^2\chi_0\tilde{\tau}_0[\eta_2 - \kappa(\theta_2 + \beta_2) + \xi_2]}{(n+1)(a - bv)},$$

$$\mu_3 = -\frac{4n^2\chi_0\tilde{\tau}_1[\eta_2 - \kappa(\theta_2 + \beta_2) + \xi_2]}{(n+1)(a - bv)},$$

$$\eta_1 = \kappa\theta_1 + \kappa\beta_1 - \xi_1 + \frac{\tilde{\tau}_1[\eta_2 - \kappa(\theta_2 + \beta_2) + \xi_2]}{\tau_1},$$

$$\mu_0 = \mu_0, \quad \tau_1 = \tau_1, \quad \tilde{\tau}_0 = \tilde{\tau}_0, \quad \tilde{\tau}_1 = \tilde{\tau}_1, \quad Q_1 = Q_2, \quad \tau_0 = \frac{\tau_1\tilde{\tau}_0}{\tilde{\tau}_1},$$

$$\omega = \frac{4n^2\chi_0\tilde{\tau}_0^{\,2}[(n+1)(\kappa(\alpha+a\kappa)+Q_2)-\tilde{\tau}_0(\eta_2 - \kappa(\theta_2+\beta_2)+\xi_2)]-\mu_0\tilde{\tau}_1^{\,2}(n+1)(a-bv)}{4n^2\chi_0\tilde{\tau}_0^{\,2}(n+1)(b\kappa-1)}. \tag{142}$$

Substituting these results into (58) and (59) leads to

$$\pm(\zeta-\zeta_0) = \sqrt{\Omega_1}\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{143}$$

where

$$\Lambda(\Psi) = \Psi^3 + \frac{\mu_2}{\mu_3}\Psi^2 + \frac{\mu_1}{\mu_3}\Psi + \frac{\mu_0}{\mu_3}, \quad \Omega_1 = \frac{\chi_0}{\mu_3}. \tag{144}$$

As a consequence, we obtain the traveling wave solutions to the coupled NLSE with power law nonlinearity as the following:

If $\Lambda(\Psi) = (\Psi-\lambda_1)^3$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{4\tau_1\Omega_1}{\left[x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t - \zeta_0\right]^2}\right\}^{\frac{1}{2n}} \times\exp\left[i\left\{-\kappa x + \left(\frac{4n^2\chi_0\tilde{\tau}_0^{\,2}[(n+1)(\kappa(\alpha+a\kappa)+Q_2)-\tilde{\tau}_0(\eta_2 - \kappa(\theta_2+\beta_2)+\xi_2)]-\mu_0\tilde{\tau}_1^{\,2}(n+1)(a-bv)}{4n^2\chi_0\tilde{\tau}_0^{\,2}(n+1)(b\kappa-1)}\right)t + \theta\right\}\right], \tag{145}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \frac{4\tilde{\tau}_1\Omega_1}{\left[x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t - \zeta_0\right]^2}\right\}^{\frac{1}{2n}} \times\exp\left[i\left\{-\kappa x + \left(\frac{4n^2\chi_0\tilde{\tau}_0^{\,2}[(n+1)(\kappa(\alpha+a\kappa)+Q_2)-\tilde{\tau}_0(\eta_2 - \kappa(\theta_2+\beta_2)+\xi_2)]-\mu_0\tilde{\tau}_1^{\,2}(n+1)(a-bv)}{4n^2\chi_0\tilde{\tau}_0^{\,2}(n+1)(b\kappa-1)}\right)t + \theta\right\}\right]. \tag{146}$$

For $\Lambda(\Psi) = (\Psi-\lambda_1)^2(\Psi-\lambda_2)$ and $\lambda_2 > \lambda_1$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_2 + \tau_1(\lambda_1-\lambda_2)\tanh^2 \times\left[\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_2}{\Omega_1}}\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t - \zeta_0\right)\right]\right\}^{\frac{1}{2n}}$$

$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}4n^2\chi_0\tilde{\tau}_0^{\,2}[(n+1)(\kappa(\alpha+a\kappa)+Q_2)-\tilde{\tau}_0(\eta_2\\ -\ \kappa(\theta_2+\beta_2)+\xi_2)]-\mu_0\tilde{\tau}_1^{\,2}(n+1)(a-bv)\end{array}}{4n^2\chi_0\tilde{\tau}_0^{\,2}(n+1)(b\kappa-1)}\right)t+\theta\right\}\right], \tag{147}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_2 + \tilde{\tau}_1(\lambda_1-\lambda_2)\tanh^2\right.$$
$$\left.\times\left[\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_2}{\Omega_1}}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t-\zeta_0\right)\right]\right\}^{\frac{1}{2n}}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}4n^2\chi_0\tilde{\tau}_0^{\,2}[(n+1)(\kappa(\alpha+a\kappa)+Q_2)-\tilde{\tau}_0(\eta_2\\ -\ \kappa(\theta_2+\beta_2)+\xi_2)]-\mu_0\tilde{\tau}_1^{\,2}(n+1)(a-bv)\end{array}}{4n^2\chi_0\tilde{\tau}_0^{\,2}(n+1)(b\kappa-1)}\right)t+\theta\right\}\right]. \tag{148}$$

However, when $\Lambda(\Psi) = (\Psi-\lambda_1)(\Psi-\lambda_2)^2$ and $\lambda_1 > \lambda_2$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \tau_1(\lambda_1-\lambda_2)\operatorname{csch}^2\left[\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_2}{\Omega_1}}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]\right\}^{\frac{1}{2n}}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}4n^2\chi_0\tilde{\tau}_0^{\,2}[(n+1)(\kappa(\alpha+a\kappa)+Q_2)-\tilde{\tau}_0(\eta_2\\ -\ \kappa(\theta_2+\beta_2)+\xi_2)]-\mu_0\tilde{\tau}_1^{\,2}(n+1)(a-bv)\end{array}}{4n^2\chi_0\tilde{\tau}_0^{\,2}(n+1)(b\kappa-1)}\right)t+\theta\right\}\right], \tag{149}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \tilde{\tau}_1(\lambda_1-\lambda_2)\operatorname{csch}^2\right.$$
$$\left.\times\left[\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_2}{\Omega_1}}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]\right\}^{\frac{1}{2n}}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}4n^2\chi_0\tilde{\tau}_0^{\,2}[(n+1)(\kappa(\alpha+a\kappa)+Q_2)-\tilde{\tau}_0(\eta_2\\ -\ \kappa(\theta_2+\beta_2)+\xi_2)]-\mu_0\tilde{\tau}_1^{\,2}(n+1)(a-bv)\end{array}}{4n^2\chi_0\tilde{\tau}_0^{\,2}(n+1)(b\kappa-1)}\right)t+\theta\right\}\right]. \tag{150}$$

On the other hand, if $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_3 + \tau_1(\lambda_2 - \lambda_3)\mathrm{sn}^2 \times \left[\mp\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{\Omega_1}}\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t - \zeta_0\right), m\right]\right\}^{\frac{1}{2n}} \times \exp\left[i\left\{-\kappa x + \left(\frac{4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha + a\kappa) + Q_2) - \tilde{\tau}_0(\eta_2 - \kappa(\theta_2 + \beta_2) + \xi_2)] - \mu_0\tilde{\tau}_1^2(n+1)(a - bv)}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{151}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_3 + \tilde{\tau}_1(\lambda_2 - \lambda_3)\mathrm{sn}^2 \times \left[\mp\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{\Omega_1}}\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t - \zeta_0\right), m\right]\right\}^{\frac{1}{2n}} \times \exp\left[i\left\{-\kappa x + \left(\frac{4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha + a\kappa) + Q_2) - \tilde{\tau}_0(\eta_2 - \kappa(\theta_2 + \beta_2) + \xi_2)] - \mu_0\tilde{\tau}_1^2(n+1)(a - bv)}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{152}$$

where

$$m^2 = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}. \tag{153}$$

It should be noted that λ_i for $i = 1, 2, 3$ are the roots of the equation

$$\Lambda(\Psi) = 0. \tag{154}$$

When $\tau_0 = -\tau_1\lambda_1$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_1$ and $\zeta_0 = 0$, the solutions (145)–(150) can be reduced to rational function solutions:

$$q(x,t) = \left\{\frac{K_2}{x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t}\right\}^{\frac{1}{n}} \times \exp\left[i\left\{-\kappa x + \left(\frac{4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha + a\kappa) + Q_2) - \tilde{\tau}_0(\eta_2 - \kappa(\theta_2 + \beta_2) + \xi_2)] - \mu_0\tilde{\tau}_1^2(n+1)(a - bv)}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{155}$$

$$r(x,t) = \left\{\frac{\tilde{K}_2}{x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1-b\kappa}\right\}t}\right\}^{\frac{1}{n}} \times \exp\left[i\left\{-\kappa x + \left(\frac{4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha+a\kappa)+Q_2) - \tilde{\tau}_0(\eta_2 - \kappa(\theta_2+\beta_2)+\xi_2)] - \mu_0\tilde{\tau}_1^2(n+1)(a-bv)}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa-1)}\right)t + \theta\right\}\right], \tag{156}$$

1-soliton solutions

$$q(x,t) = \left\{\frac{K_3}{\cosh^{\frac{1}{n}}\left[L_3\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1-b\kappa}\right\}t\right)\right]}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha+a\kappa)+Q_2) - \tilde{\tau}_0(\eta_2 - \kappa(\theta_2+\beta_2)+\xi_2)] - \mu_0\tilde{\tau}_1^2(n+1)(a-bv)}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa-1)}\right)t + \theta\right\}\right], \tag{157}$$

$$r(x,t) = \left\{\frac{\tilde{K}_3}{\cosh^{\frac{1}{n}}\left[L_3\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1-b\kappa}\right\}t\right)\right]}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha+a\kappa)+Q_2) - \tilde{\tau}_0(\eta_2 - \kappa(\theta_2+\beta_2)+\xi_2)] - \mu_0\tilde{\tau}_1^2(n+1)(a-bv)}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa-1)}\right)t + \theta\right\}\right], \tag{158}$$

and singular soliton solutions

$$q(x,t) = \left\{\frac{K_4}{\sinh^{\frac{1}{n}}\left[L_3\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1-b\kappa}\right\}t\right)\right]}\right\} \times \exp\left[i\left\{-\kappa x + \left(\frac{4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha+a\kappa)+Q_2) - \tilde{\tau}_0(\eta_2 - \kappa(\theta_2+\beta_2)+\xi_2)] - \mu_0\tilde{\tau}_1^2(n+1)(a-bv)}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa-1)}\right)t + \theta\right\}\right], \tag{159}$$

$$r(x,t) = \left\{\frac{\tilde{K}_4}{\sinh^{\frac{1}{n}}\left[L_3\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]}\right\}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha+a\kappa)+Q_2) - \tilde{\tau}_0(\eta_2\\ -\ \kappa(\theta_2+\beta_2)+\xi_2)] - \mu_0\tilde{\tau}_1^2(n+1)(a-bv)\end{array}}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa-1)}\right)t + \theta\right\}\right], \tag{160}$$

where

$$K_2 = 2\sqrt{\tau_1\Omega_1}, \quad \tilde{K}_2 = 2\sqrt{\tilde{\tau}_1\Omega_1}, \tag{161}$$

$$K_3 = [\tau_1(\lambda_2-\lambda_1)]^{\frac{1}{2n}}, \quad \tilde{K}_3 = [\tilde{\tau}_1(\lambda_2-\lambda_1)]^{\frac{1}{2n}}, \tag{162}$$

$$K_4 = [\tau_1(\lambda_1-\lambda_2)]^{\frac{1}{2n}}, \quad \tilde{K}_4 = [\tilde{\tau}_1(\lambda_1-\lambda_2)]^{\frac{1}{2n}}, \tag{163}$$

$$L_3 = \frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_2}{\Omega_1}}. \tag{164}$$

We should note that K_3, $\tilde{K}_3$ and K_4, $\tilde{K}_4$ are, respectively, the amplitudes of 1-soliton and singular soliton solutions, while L_3 is the inverse width of the solitons. These solitons are valid for $\tau_1 > 0$ and $\tilde{\tau}_1 > 0$. Furthermore, when $\tau_0 = -\tau_1\lambda_3$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_3$ and $\zeta_0 = 0$, the Jacobi elliptic function solutions (151) and (152) can be simplified as

$$q(x,t) = K_5\mathrm{sn}^{\frac{1}{n}}\left[L_j\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right), \frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right]$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha+a\kappa)+Q_2) - \tilde{\tau}_0(\eta_2\\ -\ \kappa(\theta_2+\beta_2)+\xi_2)] - \mu_0\tilde{\tau}_1^2(n+1)(a-bv)\end{array}}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa-1)}\right)t + \theta\right\}\right], \tag{165}$$

$$r(x,t) = \tilde{K}_5\mathrm{sn}^{\frac{1}{n}}\left[L_j\left(x - \left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right), \frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right]$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha+a\kappa)+Q_2) - \tilde{\tau}_0(\eta_2\\ -\ \kappa(\theta_2+\beta_2)+\xi_2)] - \mu_0\tilde{\tau}_1^2(n+1)(a-bv)\end{array}}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa-1)}\right)t + \theta\right\}\right], \tag{166}$$

where

$$K_5 = [\tau_1(\lambda_2-\lambda_3)]^{\frac{1}{2n}}, \quad \tilde{K}_5 = [\tilde{\tau}_1(\lambda_2-\lambda_3)]^{\frac{1}{2n}}, \tag{167}$$

$$L_j = \frac{(-1)^j}{2}\sqrt{\frac{\lambda_1-\lambda_3}{\Omega_1}} \quad \text{for } j=4,5. \tag{168}$$

Remark 3. When the modulus $m \to 1$, dark soliton solutions fall out

$$q(x,t) = K_5 \tanh^{\frac{1}{n}}\left[L_j\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right] \times \exp\left[i\left\{-\kappa x+\left(\frac{4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha+a\kappa)+Q_2)-\tilde{\tau}_0(\eta_2-\kappa(\theta_2+\beta_2)+\xi_2)]-\mu_0\tilde{\tau}_1^2(n+1)(a-bv)}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa-1)}\right)t+\theta\right\}\right], \tag{169}$$

$$r(x,t) = \tilde{K}_5 \tanh^{\frac{1}{n}}\left[L_j\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right] \times \exp\left[i\left\{-\kappa x+\left(\frac{4n^2\chi_0\tilde{\tau}_0^2[(n+1)(\kappa(\alpha+a\kappa)+Q_2)-\tilde{\tau}_0(\eta_2-\kappa(\theta_2+\beta_2)+\xi_2)]-\mu_0\tilde{\tau}_1^2(n+1)(a-bv)}{4n^2\chi_0\tilde{\tau}_0^2(n+1)(b\kappa-1)}\right)t+\theta\right\}\right], \tag{170}$$

where $\lambda_1 = \lambda_2$.

Case 2. If we take $\sigma = 4$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 2$ in Eq. (139), we assume that Eqs. (137) and (138) have the following formal solutions:

$$V_1 = \tau_0 + \tau_1\Psi + \tau_2\Psi^2, \tag{171}$$

$$V_2 = \tilde{\tau}_0 + \tilde{\tau}_1\Psi + \tilde{\tau}_2\Psi^2, \tag{172}$$

where τ_i and $\tilde{\tau}_i$ for $i = 0,1,2$ are constants to be determined later, and Ψ satisfies Eq. (58). Substituting these formal solutions into Eqs. (137) and (138), and solving the resulting system of algebraic equations, one recovers

$$\tau_2 = \tau_2, \quad \tilde{\tau}_0 = \tilde{\tau}_0, \quad \tilde{\tau}_1 = \tilde{\tau}_1, \quad \tilde{\tau}_2 = \tilde{\tau}_2,$$
$$Q_1 = Q_2, \quad \tau_0 = \frac{\tau_2\tilde{\tau}_0}{\tilde{\tau}_2}, \quad \tau_1 = \frac{\tau_2\tilde{\tau}_1}{\tilde{\tau}_2},$$
$$\mu_0 = -\frac{n^2\tau_2\chi_0\tilde{\tau}_0^2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1]}{\tilde{\tau}_2^2(n+1)(a-bv)},$$
$$\mu_1 = -\frac{2n^2\tau_2\chi_0\tilde{\tau}_0\tilde{\tau}_1[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1]}{\tilde{\tau}_2^2(n+1)(a-bv)},$$
$$\mu_2 = -\frac{n^2\tau_2\chi_0(\tilde{\tau}_1^2+2\tilde{\tau}_0\tilde{\tau}_2)[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1]}{\tilde{\tau}_2^2(n+1)(a-bv)},$$

$$\mu_3 = -\frac{2n^2\tau_2\chi_0\tilde{\tau}_1[\eta_1 - \kappa(\theta_1+\beta_1)+\xi_1]}{\tilde{\tau}_2(n+1)(a-bv)},$$

$$\mu_4 = -\frac{n^2\tau_2\chi_0[\eta_1 - \kappa(\theta_1+\beta_1)+\xi_1]}{(n+1)(a-bv)},$$

$$\theta_2 = \frac{\tilde{\tau}_2(\eta_2 - \kappa\beta_2+\xi_2)+\tau_2[-\eta_1+\kappa(\theta_1+\beta_1)-\xi_1]}{\kappa\tilde{\tau}_2},$$

$$\omega = \frac{\begin{array}{c}\tau_2\tilde{\tau}_1^{\,2}[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1]\\ +\ 4\tilde{\tau}_2^{\,2}(n+1)[\kappa(\alpha+a\kappa)+Q_2]\end{array}}{4\tilde{\tau}_2^{\,2}(n+1)(b\kappa-1)}. \tag{173}$$

Substituting the solution set (173) into Eqs. (58) and (59) leads to

$$\pm(\zeta-\zeta_0) = \Omega_2\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{174}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4}, \quad \Omega_2 = \sqrt{\frac{\chi_0}{\mu_4}}. \tag{175}$$

Integrating (174) and taking $\zeta_0 = 0$, we have the traveling wave solutions to (102) and (103) in the following forms:

For $\Lambda(\Psi) = (\Psi-\lambda_1)^4$,

$$q(x,t) = \left[\sum_{j=0}^{2}\tau_j\left(\lambda_1 \pm \frac{\Omega_2}{x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t}\right)^j\right]^{\frac{1}{2n}} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}\tau_2\tilde{\tau}_1^{\,2}[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1]\\ -4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1]\\ +4\tilde{\tau}_2^{\,2}(n+1)[\kappa(\alpha+a\kappa)+Q_2]\end{array}}{4\tilde{\tau}_2^{\,2}(n+1)(b\kappa-1)}\right)t+\theta\right\}\right], \tag{176}$$

$$r(x,t) = \left[\sum_{j=0}^{2}\tilde{\tau}_j\left(\lambda_1 \pm \frac{\Omega_2}{x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t}\right)^j\right]^{\frac{1}{2n}} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{c}\tau_2\tilde{\tau}_1^{\,2}[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1]\\ -4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1]\\ +4\tilde{\tau}_2^{\,2}(n+1)[\kappa(\alpha+a\kappa)+Q_2]\end{array}}{4\tilde{\tau}_2^{\,2}(n+1)(b\kappa-1)}\right)t+\theta\right\}\right]. \tag{177}$$

If $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$,

$$q(x,t) = \left[\sum_{j=0}^{2} \tau_j \left(\lambda_1 + \frac{4\Omega_2^2(\lambda_2 - \lambda_1)}{4\Omega_2^2 - \left[(\lambda_1 - \lambda_2)\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right)\right]^2}\right)^j\right]^{\frac{1}{2n}} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{l}\tau_2\tilde{\tau}_1^2[\eta_1 - \kappa(\theta_1 + \beta_1) + \xi_1] \\ \quad - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1 - \kappa(\theta_1 + \beta_1) + \xi_1] \\ \quad + 4\tilde{\tau}_2^2(n+1)[\kappa(\alpha + a\kappa) + Q_2]\end{array}}{4\tilde{\tau}_2^2(n+1)(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{178}$$

$$r(x,t) = \left[\sum_{j=0}^{2} \tilde{\tau}_j \left(\lambda_1 + \frac{4\Omega_2^2(\lambda_2 - \lambda_1)}{4\Omega_2^2 - \left[(\lambda_1 - \lambda_2)\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right)\right]^2}\right)^j\right]^{\frac{1}{2n}} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{l}\tau_2\tilde{\tau}_1^2[\eta_1 - \kappa(\theta_1 + \beta_1) + \xi_1] \\ \quad - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1 - \kappa(\theta_1 + \beta_1) + \xi_1] \\ \quad + 4\tilde{\tau}_2^2(n+1)[\kappa(\alpha + a\kappa) + Q_2]\end{array}}{4\tilde{\tau}_2^2(n+1)(b\kappa - 1)}\right)t + \theta\right\}\right]. \tag{179}$$

However, when $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$,

$$q(x,t) = \left[\sum_{j=0}^{2} \tau_j \left(\lambda_2 + \frac{\lambda_2 - \lambda_1}{\exp\left[\frac{\lambda_1 - \lambda_2}{\Omega_2}\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right)\right] - 1}\right)^j\right]^{\frac{1}{2n}} \times \exp\left[i\left\{-\kappa x + \left(\frac{\begin{array}{l}\tau_2\tilde{\tau}_1^2[\eta_1 - \kappa(\theta_1 + \beta_1) + \xi_1] \\ \quad - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1 - \kappa(\theta_1 + \beta_1) + \xi_1] \\ \quad + 4\tilde{\tau}_2^2(n+1)[\kappa(\alpha + a\kappa) + Q_2]\end{array}}{4\tilde{\tau}_2^2(n+1)(b\kappa - 1)}\right)t + \theta\right\}\right], \tag{180}$$

$$r(x,t) = \left[\sum_{j=0}^{2} \tilde{\tau}_j \left(\lambda_2 + \frac{\lambda_2 - \lambda_1}{\exp\left[\frac{\lambda_1 - \lambda_2}{\Omega_2}\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right)\right] - 1}\right)^j\right]^{\frac{1}{2n}}$$

$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_2\tilde{\tau}_1^{\,2}[\eta_1 - \kappa(\theta_1+\beta_1)+\xi_1] - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] + 4\tilde{\tau}_2^{\,2}(n+1)[\kappa(\alpha+a\kappa)+Q_2]}{4\tilde{\tau}_2^{\,2}(n+1)(b\kappa-1)}\right)t+\theta\right\}\right], \tag{181}$$

and

$$q(x,t) = \left[\sum_{j=0}^{2}\tau_j\left(\lambda_1 + \frac{\lambda_1-\lambda_2}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega_2}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]-1}\right)^j\right]^{\frac{1}{2n}} \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_2\tilde{\tau}_1^{\,2}[\eta_1 - \kappa(\theta_1+\beta_1)+\xi_1] - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] + 4\tilde{\tau}_2^{\,2}(n+1)[\kappa(\alpha+a\kappa)+Q_2]}{4\tilde{\tau}_2^{\,2}(n+1)(b\kappa-1)}\right)t+\theta\right\}\right], \tag{182}$$

$$r(x,t) = \left[\sum_{j=0}^{2}\tilde{\tau}_j\left(\lambda_1 + \frac{\lambda_1-\lambda_2}{\exp\left[\frac{\lambda_1-\lambda_2}{\Omega_2}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]-1}\right)^j\right]^{\frac{1}{2n}} \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_2\tilde{\tau}_1^{\,2}[\eta_1 - \kappa(\theta_1+\beta_1)+\xi_1] - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] + 4\tilde{\tau}_2^{\,2}(n+1)[\kappa(\alpha+a\kappa)+Q_2]}{4\tilde{\tau}_2^{\,2}(n+1)(b\kappa-1)}\right)t+\theta\right\}\right]. \tag{183}$$

Whenever $\Lambda(\Psi) = (\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$,

$$q(x,t) = \left[\sum_{j=0}^{2}\tau_j\left(\lambda_1 - \frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh \times \left[\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{\Omega_2}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]}\right)^j\right]^{\frac{1}{2n}}$$

$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_2\tilde{\tau}_1^2[\eta_1 - \kappa(\theta_1+\beta_1)+\xi_1] - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1 - \kappa(\theta_1+\beta_1)+\xi_1] + 4\tilde{\tau}_2^2(n+1)[\kappa(\alpha+a\kappa)+Q_2]}{4\tilde{\tau}_2^2(n+1)(b\kappa-1)}\right)t+\theta\right\}\right], \tag{184}$$

$$r(x,t) = \left[\sum_{j=0}^{2}\tilde{\tau}_j\left(\lambda_1 - \frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh \times \left[\frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{\Omega_2}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]}\right)^j\right]^{\frac{1}{2n}}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_2\tilde{\tau}_1^2[\eta_1 - \kappa(\theta_1+\beta_1)+\xi_1] - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1 - \kappa(\theta_1+\beta_1)+\xi_1] + 4\tilde{\tau}_2^2(n+1)[\kappa(\alpha+a\kappa)+Q_2]}{4\tilde{\tau}_2^2(n+1)(b\kappa-1)}\right)t+\theta\right\}\right]. \tag{185}$$

Finally, if $\Lambda(\Psi) = (\Psi-\lambda_1)(\Psi-\lambda_2)(\Psi-\lambda_3)(\Psi-\lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$,

$$q(x,t) = \left[\sum_{j=0}^{2}\tau_j\left(\lambda_2 + \frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\mathrm{sn}^2 \times \left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega_2}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right), m\right]}\right)^j\right]^{\frac{1}{2n}}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_2\tilde{\tau}_1^2[\eta_1 - \kappa(\theta_1+\beta_1)+\xi_1] - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1 - \kappa(\theta_1+\beta_1)+\xi_1] + 4\tilde{\tau}_2^2(n+1)[\kappa(\alpha+a\kappa)+Q_2]}{4\tilde{\tau}_2^2(n+1)(b\kappa-1)}\right)t+\theta\right\}\right], \tag{186}$$

$$r(x,t) = \left[\sum_{j=0}^{2} \tilde{\tau}_j \left(\lambda_2 + \frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\mathrm{sn}^2 \times \left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega_2}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right), m\right]} \right)^j \right]^{\frac{1}{2n}}$$

$$\times \exp\left[i\left\{ -\kappa x + \left(\frac{\begin{array}{l}\tau_2\tilde{\tau}_1^{\,2}[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] \\ \quad -4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] \\ \quad +4\tilde{\tau}_2^{\,2}(n+1)[\kappa(\alpha+a\kappa)+Q_2]\end{array}}{4\tilde{\tau}_2^{\,2}(n+1)(b\kappa-1)} \right) t + \theta \right\} \right], \tag{187}$$

where

$$m^2 = \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \tag{188}$$

It should be noted that λ_i for $i = 1, \ldots, 4$ are the roots of the equation

$$\Lambda(\Psi) = 0. \tag{189}$$

Remark 4. When the modulus $m \to 1$, hyperbolic function solutions fall out:

$$q(x,t) = \left[\sum_{j=0}^{2} \tau_j \left(\lambda_2 + \frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\tanh^2 \times \left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega_2}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]} \right)^j \right]^{\frac{1}{2n}}$$

$$\times \exp\left[i\left\{ -\kappa x + \left(\frac{\begin{array}{l}\tau_2\tilde{\tau}_1^{\,2}[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] \\ \quad -4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] \\ \quad +4\tilde{\tau}_2^{\,2}(n+1)[\kappa(\alpha+a\kappa)+Q_2]\end{array}}{4\tilde{\tau}_2^{\,2}(n+1)(b\kappa-1)} \right) t + \theta \right\} \right], \tag{190}$$

$$r(x,t) = \left[\sum_{j=0}^{2} \tilde{\tau}_j \left(\lambda_2 + \frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\tanh^2 \times \left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega_2}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]}\right)^j\right]^{\frac{1}{2n}}$$

$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_2\tilde{\tau}_1^2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] + 4\tilde{\tau}_2^2(n+1)[\kappa(\alpha+a\kappa)+Q_2]}{4\tilde{\tau}_2^2(n+1)(b\kappa-1)}\right)t+\theta\right\}\right], \quad (191)$$

where $\lambda_3 = \lambda_4$.

Remark 5. However, if $m \to 0$, periodic wave solutions are acquired:

$$q(x,t) = \left[\sum_{j=0}^{2} \tau_j \left(\lambda_2 + \frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\sin^2 \times \left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega_2}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]}\right)^j\right]^{\frac{1}{2n}}$$

$$\times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_2\tilde{\tau}_1^2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] - 4\tau_2\tilde{\tau}_0\tilde{\tau}_2[\eta_1-\kappa(\theta_1+\beta_1)+\xi_1] + 4\tilde{\tau}_2^2(n+1)[\kappa(\alpha+a\kappa)+Q_2]}{4\tilde{\tau}_2^2(n+1)(b\kappa-1)}\right)t+\theta\right\}\right], \quad (192)$$

$$r(x,t) = \left[\sum_{j=0}^{2} \tilde{\tau}_j \left(\lambda_2 + \frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\sin^2 \times \left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2\Omega_2}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right]}\right)^j\right]^{\frac{1}{2n}}$$

$$\times \exp\left[i \left\{ -\kappa x + \left(\frac{\begin{array}{l} \tau_2 \tilde{\tau}_1^2[\eta_1 - \kappa(\theta_1 + \beta_1) + \xi_1] \\ \quad - 4\tau_2 \tilde{\tau}_0 \tilde{\tau}_2 [\eta_1 - \kappa(\theta_1 + \beta_1) + \xi_1] \\ \quad + 4\tilde{\tau}_2^2 (n+1)[\kappa(\alpha + a\kappa) + Q_2] \end{array}}{4\tilde{\tau}_2^2 (n+1)(b\kappa - 1)} \right) t + \theta \right\} \right], \tag{193}$$

where $\lambda_2 = \lambda_3$.

4. Log Law

For log-law nonlinear media, $F(s) = \ln s$, solitons in magneto-optic waveguides are modeled by

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + \{\xi_1 \ln |q|^2 + \eta_1 \ln |r|^2\} q = Q_1 r + i\alpha_1 q_x, \tag{194}$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + \{\xi_2 \ln |r|^2 + \eta_2 \ln |q|^2\} r = Q_2 q + i\alpha_2 r_x. \tag{195}$$

For this kind of nonlinearity, substituting (5) and (6) into (194) and (195) and then decomposing into real and imaginary parts give

$$(a_l - b_l v) P_l'' + (b_l \omega \kappa - \omega - a_l \kappa^2 - \alpha_l \kappa) P_l + 2\xi_l P_l \ln P_l + 2\eta_l P_l \ln P_{\bar{l}} = Q_l P_{\bar{l}}, \tag{196}$$

and

$$v(b_l \kappa - 1) P_l' + (b_l \omega - 2a_l \kappa - \alpha_l) P_l' = 0, \tag{197}$$

respectively. From (197), it is possible to retrieve the solitons speed (11) as long as the constraint (12). Consequently, (14) and (15) are also satisfied in this case, and the real part (196) becomes

$$(a - bv) P_l'' + (b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa) P_l + 2\xi_l P_l \ln P_l + 2\eta_l P_l \ln P_{\bar{l}} = Q_l P_{\bar{l}}. \tag{198}$$

To obtain an analytic solution, we use the transformations

$$P_1 = \exp \frac{1}{V_1} = \exp \frac{1}{V_2} = P_2 \tag{199}$$

in Eq. (198) to find

$$\begin{aligned} &(a - bv)\{(V_1')^2 + 2V_1 (V_1')^2 - V_1^2 V_1''\} + 2(\xi_1 + \eta_1) V_1^3 \\ &\quad + (b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa - Q_1) V_1^4 = 0, \end{aligned} \tag{200}$$

$$\begin{aligned} &(a - bv)\{(V_2')^2 + 2V_2 (V_2')^2 - V_2^2 V_2''\} + 2(\xi_2 + \eta_2) V_2^3 \\ &\quad + (b\omega\kappa - \omega - a\kappa^2 - \alpha\kappa - Q_2) V_2^4 = 0. \end{aligned} \tag{201}$$

For log-law nonlinear medium, the modified simple equation method and trial equation method are not applicable.

4.1. *Extended trial function method*

The extended trial function scheme will be applied to Eqs. (200) and (201) to retrieve bright, dark, and singular solitons. According to the balance principle, we determine a relation of σ, ρ, ς and $\tilde{\varsigma}$ as

$$\varsigma = \tilde{\varsigma} = \sigma - \rho - 2. \tag{202}$$

When $\sigma = 3$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (202), Eqs. (200) and (201) have the solutions in the forms

$$V_1 = \tau_0 + \tau_1 \Psi, \tag{203}$$

$$V_2 = \tilde{\tau}_0 + \tilde{\tau}_1 \Psi, \tag{204}$$

where τ_i and $\tilde{\tau}_i$ for $i = 0, 1$ are constants to be determined later, and Ψ satisfies Eq. (58). Substituting these formal solutions into Eqs. (200) and (201), and solving the resulting system of algebraic equations, we have

$$\begin{aligned}
\chi_0 &= -\frac{\mu_2(a - bv)(\tau_1 - \tilde{\tau}_1)}{6\tau_1\tilde{\tau}_0(Q_1 - Q_2)},\\
\eta_1 &= \frac{\tilde{\tau}_1(\xi_1 + Q_1 - Q_2) - \xi_1\tau_1}{\tau_1 - \tilde{\tau}_1},\\
\eta_2 &= \frac{\tau_1(Q_1 - Q_2) - \xi_2(\tau_1 - \tilde{\tau}_1)}{\tau_1 - \tilde{\tau}_1},\\
\tau_1 &= \tau_1, \quad \tilde{\tau}_0 = \tilde{\tau}_0, \quad \tilde{\tau}_1 = \tilde{\tau}_1, \quad \tau_0 = \frac{\tau_1\tilde{\tau}_0}{\tilde{\tau}_1},\\
\mu_2 &= \mu_2, \quad \mu_0 = \frac{\mu_2\tilde{\tau}_0^{\,2}}{3\tilde{\tau}_1^{\,2}}, \quad \mu_1 = \frac{\mu_2\tilde{\tau}_0}{\tilde{\tau}_1}, \quad \mu_3 = \frac{\mu_2\tilde{\tau}_1}{3\tilde{\tau}_0},\\
\omega &= \frac{\tau_1[\kappa(\alpha + a\kappa) + Q_1] - \tilde{\tau}_1[\kappa(\alpha + a\kappa) + Q_2]}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}.
\end{aligned} \tag{205}$$

Substituting these results into Eqs. (58) and (59) leads to

$$\pm(\zeta - \zeta_0) = \sqrt{\Omega_3}\int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{206}$$

where

$$\Lambda(\Psi) = \Psi^3 + \frac{\mu_2}{\mu_3}\Psi^2 + \frac{\mu_1}{\mu_3}\Psi + \frac{\mu_0}{\mu_3}, \quad \Omega_3 = \frac{\chi_0}{\mu_3}. \tag{207}$$

Consequently, we obtain the traveling wave solutions to Eqs. (194) and (195) in the following forms:

For $\Lambda(\Psi) = (\Psi - \lambda_1)^3$,

$$\begin{aligned}
q(x,t) &= \exp\left[\tau_0 + \tau_1\lambda_1 + \frac{4\tau_1\Omega_3}{\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t - \zeta_0\right)^2}\right]^{-1}\\
&\quad \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_1[\kappa(\alpha + a\kappa) + Q_1] - \tilde{\tau}_1[\kappa(\alpha + a\kappa) + Q_2]}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}\right)t + \theta\right\}\right],
\end{aligned} \tag{208}$$

$$r(x,t) = \exp\left[\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \frac{4\tilde{\tau}_1\Omega_3}{\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t - \zeta_0\right)^2}\right]^{-1} \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_1[\kappa(\alpha + a\kappa) + Q_1] - \tilde{\tau}_1[\kappa(\alpha + a\kappa) + Q_2]}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}\right)t + \theta\right\}\right]. \quad (209)$$

If $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$,

$$q(x,t) = \exp\left[\tau_0 + \tau_1\lambda_2 + \tau_1(\lambda_1 - \lambda_2) \times \tanh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{\Omega_3}}\left[x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t - \zeta_0\right]\right)\right]^{-1} \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_1[\kappa(\alpha + a\kappa) + Q_1] - \tilde{\tau}_1[\kappa(\alpha + a\kappa) + Q_2]}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}\right)t + \theta\right\}\right], \quad (210)$$

$$r(x,t) = \exp\left[\tilde{\tau}_0 + \tilde{\tau}_1\lambda_2 + \tilde{\tau}_1(\lambda_1 - \lambda_2) \times \tanh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{\Omega_3}}\left[x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t - \zeta_0\right]\right)\right]^{-1} \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_1[\kappa(\alpha + a\kappa) + Q_1] - \tilde{\tau}_1[\kappa(\alpha + a\kappa) + Q_2]}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}\right)t + \theta\right\}\right]. \quad (211)$$

However, when $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)^2$ and $\lambda_1 > \lambda_2$,

$$q(x,t) = \exp\left[\tau_0 + \tau_1\lambda_1 + \tau_1(\lambda_1 - \lambda_2)\operatorname{csch}^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{\Omega_3}}\left[x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right]\right)\right]^{-1} \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_1[\kappa(\alpha + a\kappa) + Q_1] - \tilde{\tau}_1[\kappa(\alpha + a\kappa) + Q_2]}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}\right)t + \theta\right\}\right], \quad (212)$$

$$r(x,t) = \exp\left[\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \tilde{\tau}_1(\lambda_1 - \lambda_2)\operatorname{csch}^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{\Omega_3}}\left[x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right]\right)\right]^{-1} \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_1[\kappa(\alpha + a\kappa) + Q_1] - \tilde{\tau}_1[\kappa(\alpha + a\kappa) + Q_2]}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}\right)t + \theta\right\}\right]. \quad (213)$$

On the other hand, if $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$,

$$q(x,t) = \exp\left[\tau_0 + \tau_1\lambda_3 + \tau_1(\lambda_2 - \lambda_3)\mathrm{sn}^2 \times \left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{\Omega_3}}\left[x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t - \zeta_0\right], m\right)\right]^{-1} \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_1[\kappa(\alpha + a\kappa) + Q_1] - \tilde{\tau}_1[\kappa(\alpha + a\kappa) + Q_2]}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}\right)t + \theta\right\}\right], \tag{214}$$

$$r(x,t) = \exp\left[\tilde{\tau}_0 + \tilde{\tau}_1\lambda_3 + \tilde{\tau}_1(\lambda_2 - \lambda_3)\mathrm{sn}^2 \times \left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{\Omega_3}}\left[x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t - \zeta_0\right], m\right)\right]^{-1} \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_1[\kappa(\alpha + a\kappa) + Q_1] - \tilde{\tau}_1[\kappa(\alpha + a\kappa) + Q_2]}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}\right)t + \theta\right\}\right], \tag{215}$$

where

$$m^2 = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}. \tag{216}$$

It should be noted that λ_i for $i = 1, 2, 3$ are the roots of the equation

$$\Lambda(\Psi) = 0. \tag{217}$$

Under the conditions $\tau_0 = -\tau_1\lambda_1$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_1$ and $\zeta_0 = 0$, solutions (208)–(213) are reduced to exact solutions as the following:

$$q(x,t) = \exp\left[\frac{\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right)^2}{4\tau_1\Omega_3}\right] \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_1[\kappa(\alpha + a\kappa) + Q_1] - \tilde{\tau}_1[\kappa(\alpha + a\kappa) + Q_2]}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}\right)t + \theta\right\}\right], \tag{218}$$

$$r(x,t) = \exp\left[\frac{\left(x - \left\{\frac{b\omega - 2a\kappa - \alpha}{1 - b\kappa}\right\}t\right)^2}{4\tilde{\tau}_1\Omega_3}\right] \times \exp\left[i\left\{-\kappa x + \left(\frac{\tau_1[\kappa(\alpha + a\kappa) + Q_1] - \tilde{\tau}_1[\kappa(\alpha + a\kappa) + Q_2]}{(b\kappa - 1)(\tau_1 - \tilde{\tau}_1)}\right)t + \theta\right\}\right], \tag{219}$$

$$q(x,t) = \exp\left[\frac{1}{\tau_1(\lambda_2-\lambda_1)}\cosh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_2}{\Omega_3}}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right)\right] \times \exp\left[i\left\{-\kappa x+\left(\frac{\tau_1[\kappa(\alpha+a\kappa)+Q_1]-\tilde{\tau}_1[\kappa(\alpha+a\kappa)+Q_2]}{(b\kappa-1)(\tau_1-\tilde{\tau}_1)}\right)t+\theta\right\}\right], \tag{220}$$

$$r(x,t) = \exp\left[\frac{1}{\tilde{\tau}_1(\lambda_2-\lambda_1)}\cosh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_2}{\Omega_3}}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right)\right] \times \exp\left[i\left\{-\kappa x+\left(\frac{\tau_1[\kappa(\alpha+a\kappa)+Q_1]-\tilde{\tau}_1[\kappa(\alpha+a\kappa)+Q_2]}{(b\kappa-1)(\tau_1-\tilde{\tau}_1)}\right)t+\theta\right\}\right], \tag{221}$$

and

$$q(x,t) = \exp\left[\frac{1}{\tau_1(\lambda_1-\lambda_2)}\sinh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_2}{\Omega_3}}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right)\right] \times \exp\left[i\left\{-\kappa x+\left(\frac{\tau_1[\kappa(\alpha+a\kappa)+Q_1]-\tilde{\tau}_1[\kappa(\alpha+a\kappa)+Q_2]}{(b\kappa-1)(\tau_1-\tilde{\tau}_1)}\right)t+\theta\right\}\right], \tag{222}$$

$$r(x,t) = \exp\left[\frac{1}{\tilde{\tau}_1(\lambda_1-\lambda_2)}\sinh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_2}{\Omega_3}}\left(x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right)\right)\right] \times \exp\left[i\left\{-\kappa x+\left(\frac{\tau_1[\kappa(\alpha+a\kappa)+Q_1]-\tilde{\tau}_1[\kappa(\alpha+a\kappa)+Q_2]}{(b\kappa-1)(\tau_1-\tilde{\tau}_1)}\right)t+\theta\right\}\right]. \tag{223}$$

Moreover, if $\tau_0 = -\tau_1\lambda_3$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_3$ and $\zeta_0 = 0$, the solutions (214) and (215) are transformed to

$$q(x,t) = \exp\left[\frac{1}{\tau_1(\lambda_2-\lambda_3)}\mathrm{ns}^2\left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_3}{\Omega_3}}\left[x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right],\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right)\right] \times \exp\left[i\left\{-\kappa x+\left(\frac{\tau_1[\kappa(\alpha+a\kappa)+Q_1]-\tilde{\tau}_1[\kappa(\alpha+a\kappa)+Q_2]}{(b\kappa-1)(\tau_1-\tilde{\tau}_1)}\right)t+\theta\right\}\right], \tag{224}$$

$$r(x,t) = \exp\left[\frac{1}{\tilde{\tau}_1(\lambda_2-\lambda_3)}\mathrm{ns}^2\left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_3}{\Omega_3}}\left[x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right],\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right)\right] \times \exp\left[i\left\{-\kappa x+\left(\frac{\tau_1[\kappa(\alpha+a\kappa)+Q_1]-\tilde{\tau}_1[\kappa(\alpha+a\kappa)+Q_2]}{(b\kappa-1)(\tau_1-\tilde{\tau}_1)}\right)t+\theta\right\}\right]. \tag{225}$$

Remark 6. When the modulus $m \to 1$, the following solutions emerge:

$$q(x,t) = \exp\left[\frac{1}{\tau_1(\lambda_2-\lambda_3)}\coth^2\left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_3}{\Omega_3}}\left[x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right]\right)\right] \times \exp\left[i\left\{-\kappa x+\left(\frac{\tau_1[\kappa(\alpha+a\kappa)+Q_1]-\tilde{\tau}_1[\kappa(\alpha+a\kappa)+Q_2]}{(b\kappa-1)(\tau_1-\tilde{\tau}_1)}\right)t+\theta\right\}\right], \tag{226}$$

$$r(x,t) = \exp\left[\frac{1}{\tilde{\tau}_1(\lambda_2-\lambda_3)}\coth^2\left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1-\lambda_3}{\Omega_3}}\left[x-\left\{\frac{b\omega-2a\kappa-\alpha}{1-b\kappa}\right\}t\right]\right)\right] \times \exp\left[i\left\{-\kappa x+\left(\frac{\tau_1[\kappa(\alpha+a\kappa)+Q_1]-\tilde{\tau}_1[\kappa(\alpha+a\kappa)+Q_2]}{(b\kappa-1)(\tau_1-\tilde{\tau}_1)}\right)t+\theta\right\}\right], \tag{227}$$

where $\lambda_1 = \lambda_2$.

5. Conclusions

This paper obtained bright, dark, and singular soliton solutions in magneto-optic waveguides where soliton perturbation theory was studied. The third form of nonlinear medium that was studied in this paper is the log-law nonlinearity where optical Gaussons were retrieved. These soliton and Gausson solutions all appear with constraint conditions that guarantee their existence. These magneto-optic solitons will be of great advantage to pulse transmission along optical fibers that will provide the state of soliton separation rather than the state of soliton "clutter". Additionally, it is possible to steer nematicons in liquid crystals with such magneto-optic control.[11]

The results of this paper stand on a strong footing for further advancements in this area. In future, additional laws of nonlinear media will be studied. These include anti-cubic nonlinearity, quadratic-cubic nonlinearity, parabolic law, dual-power law, quadratic–cubic-septic law, triple power law, and many others. Those results will be published elsewhere sequentially.

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