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ELECTRODYNAMICS AND WAVE PROPAGATION

Cubic–Quartic Optical Solitons with Differential Group Delay for Kudryashov's Model by Extended Trial Function

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Abstract—This paper implements mathematically rigorous extended trial function algorithm to address cubic–quartic optical solitons in birefringent fibers having Kudryashov's law of nonlinear refractive index. Three special cases of the power-law nonlinearity parameter are taken into consideration. Bright and singular optical solitons emerge from this analytical scheme.

Keywords: cubic–quartic solitons, birefringence, Kudryashov's refractive index, extended trial function

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1. INTRODUCTION

Two important concepts were developed during the past couple of years. They are the cubic–quartic (CQ) solitons [3, 9] and Kudryashov's equation, namely Kudryashov's law of nonlinear refractive index [1, 4–7, 10]. CQ solitons is an extension of pure-quartic solitons that was first proposed during 2016 [3]. Subsequently, the concept of CQ solitons was merged with Kudryashov's law and consequently the concept of CQ optical solitons with Kudryashov's law of refractive index was conceived. Some preliminary results were reported from this new concept for polarization-preserving optical fibers only [1]. The current paper extends the same dynamics to birefringent fibers for polarization-mode dispersion. There are three cases for the power-law nonlinearity parameter, in Kudryashov's form of refractive index, that are taken into account. These three parameter values are within the domain of existence of the solitons as reported in earlier works. The mathematically rigorous extended trial function scheme is applied to all of the three cases successfully to reveal soliton solutions to the model in birefringent fibers. The results are all enumerated in the subsequent sections after an introductory discussion.

1.1. Governing Model

The governing CQ equation with Kudryashov's form of nonlinear refractive index in polarization-preserving fiber is given below [1]:

$$iq_t + iaq_{xxx} + bq_{xxxx} + \left(\frac{c_1}{|q|^{2n}} + \frac{c_2}{|q|^n} + c_3|q|^n + c_4|q|^{2n}\right)q = 0, \tag{1}$$

with $i = \sqrt{-1}$, where the first term is the linear temporal evolution, while a represents third-order dispersion (3OD) coefficient and b is the coefficient of fourth–order dispersion (4OD). The constant coefficients c_j for $j = 1, 2, 3, 4$ give self-phase modulation (SPM) effect. The next subsections will introduce the model in birefringent fibers with three cases at $n = 1$, $n = 2$ and $n = 3$. The details are given in the next three subsections.

1.1.1. Case 1: ($n = 1$). For optical fibers with differential group delay, KE (1) splits into two components due to birefringence at $n = 1$. Then, the vector-coupled KE reads

$$iu_t + ia_1u_{xxx} + b_1u_{xxxx} + \frac{p_1u}{c_1|u|^2 + d_1|v|^2} + \frac{q_1u}{\sqrt{|u|^2+|v|^2}} + r_1u\sqrt{|u|^2+|v|^2} + \left(\alpha_1|u|^2 + \beta_1|v|^2\right)u = 0, \tag{2}$$

$$iv_t + ia_2v_{xxx} + b_2v_{xxxx} + \frac{p_2v}{c_2|v|^2 + d_2|u|^2} + \frac{q_2v}{\sqrt{|v|^2 + |u|^2}} + r_2v\sqrt{|v|^2 + |u|^2} + (\alpha_2|v|^2 + \beta_2|u|^2)v = 0, \quad (3)$$

where a_l, b_l, c_l, d_l, p_l, q_l, r_l, α_l and β_l for $l = 1, 2$ are constants, while the independent variables x and t stand for spatial and temporal variables, respectively and the dependent variables $u(x, t)$ and $v(x, t)$ are wave profiles along the two components. The coefficients a_l and b_l are real parameters that independently controls 3OD and 4OD respectively, while the coefficients p_l, q_l and r_l represent the combination of SPM and cross phase modulation effects (XPM). The coefficients c_l and α_l give SPM and the coefficients d_l and β_l are XPM, respectively.

1.1.2. Case 2: ($n = 2$). For optical fibers with differential group delay, KE splits into two components from the effect of birefringence at $n = 2$. In this case, the governing coupled KE is given by

$$iu_t + ia_1u_{xxx} + b_1u_{xxxx} + \frac{p_1u}{c_1|u|^4 + d_1|u|^2|v|^2 + e_1|v|^4} + \frac{q_1u}{f_1|u|^2 + g_1|v|^2} + (\alpha_1|u|^2 + \beta_1|v|^2)u + (\xi_1|u|^4 + \eta_1|u|^2|v|^2 + \zeta_1|v|^4)u = 0, \quad (4)$$

$$iv_t + ia_2v_{xxx} + b_2v_{xxxx} + \frac{p_2v}{c_2|v|^4 + d_2|v|^2|u|^2 + e_2|u|^4} + \frac{q_2v}{f_2|v|^2 + g_2|u|^2} + (\alpha_2|v|^2 + \beta_2|u|^2)v + (\xi_2|v|^4 + \eta_2|v|^2|u|^2 + \zeta_2|u|^4)v = 0, \quad (5)$$

where a_l, b_l, c_l, d_l, e_l, f_l, g_l, p_l, q_l, α_l, β_l, ξ_l, η_l and ζ_l for $l = 1, 2$ are constants. The coefficients a_l and b_l are real parameters that independently controls 3OD and 4OD respectively, while the coefficients p_l, q_l and η_l represent the combination of SPM and XPM. Also, the coefficients c_l, f_l, α_l and ξ_l are SPM, while the coefficients d_l, e_l, g_l, β_l and ζ_l are from XPM, respectively.

1.1.3. Case 3: ($n = 3$). For optical fibers with differential group delay, KE splits into two components because of birefringence at $n = 3$. Thus, KE in birefringent fibers is

$$iu_t + ia_1u_{xxx} + b_1u_{xxxx} + \frac{p_1u}{c_1|u|^6 + d_1|u|^4|v|^2 + e_1|v|^4|u|^2 + f_1|v|^6} + \frac{q_1u}{(g_1|u|^2 + h_1|v|^2)\sqrt{|u|^2 + |v|^2}} + u(\alpha_1|u|^2 + \beta_1|v|^2)\sqrt{|u|^2 + |v|^2} + (\xi_1|u|^6 + \eta_1|u|^4|v|^2 + \zeta_1|v|^4|u|^2 + \theta_1|v|^6)u = 0, \quad (6)$$

$$iv_t + ia_2v_{xxx} + b_2v_{xxxx} + \frac{p_2v}{c_2|v|^6 + d_2|v|^4|u|^2 + e_2|u|^4|v|^2 + f_2|u|^6} + \frac{q_2v}{(g_2|v|^2 + h_2|u|^2)\sqrt{|v|^2 + |u|^2}} + v(\alpha_2|v|^2 + \beta_2|u|^2)\sqrt{|v|^2 + |u|^2} + (\xi_2|v|^6 + \eta_2|v|^4|u|^2 + \zeta_2|u|^4|v|^2 + \theta_2|u|^6)v = 0, \quad (7)$$

where a_l, b_l, c_l, d_l, e_l, f_l, g_l, h_l, p_l, q_l, α_l, β_l, ξ_l, η, ζ and θ_l for $l = 1, 2$ are constants. The coefficients a_l and b_l are real parameters that independently controls 3OD and 4OD respectively, while the coefficients g_l, h_l, p_l, q_l, α_l, β_l, η_l, ζ_l stand for the combination of SPM and XPM. Also, the coefficients c_l and ξ_l are SPM, while the coefficients d_l, e_l, f_l and θ_l are XPM, respectively.

2. MATHEMATICAL ANALYSIS

2.1. Case 1: (n = 1)

To kick off, the initial hypothesis is selected as [2]:

$$u(x,t) = P_1(s)\exp(i\phi), \quad (8)$$

$$v(x,t) = P_2(s)\exp(i\phi), \quad (9)$$

where

$$s = x - vt, \quad (10)$$

and v the speed of the soliton. The phase ϕ has the form:

$$\phi = -\kappa x + \omega t + \theta. \quad (11)$$

Here, κ is the frequency, ω is the wave number and ω is the phase constant. Substitute (8) and (9) into (2) and (3). Then, real parts give

$$3a_1\kappa P_1'' + b_1\left(P_1^{(4)} - 6\kappa^2P_1''\right) + \left(b_1\kappa^4 - \omega - a_1\kappa^3\right)P_1 + \frac{p_1P_1}{c_1P_1^2 + d_1P_2^2} + \frac{q_1P_1}{\sqrt{P_1^2 + P_2^2}} + r_1P_1\sqrt{P_1^2 + P_2^2} + \left(\alpha_1P_1^2 + \beta_1P_2^2\right)P_1 = 0, \quad (12)$$

$$3a_2\kappa P_2'' + b_2\left(P_2^{(4)} - 6\kappa^2P_2''\right) + \left(b_2\kappa^4 - \omega - a_2\kappa^3\right)P_2 + \frac{p_2P_2}{c_2P_2^2 + d_2P_1^2} + \frac{q_2P_2}{\sqrt{P_1^2 + P_2^2}} + r_2P_2\sqrt{P_1^2 + P_2^2} + \left(\alpha_2P_2^2 + \beta_2P_2^2\right)P_2 = 0, \quad (13)$$

while imaginary parts are

$$\left(v + 3a_1\kappa^2 - 4b_1\kappa^3\right)P_1' - (a_1 - 4b_1\kappa)P_1^{(3)} = 0, \quad (14)$$

$$\left(v + 3a_2\kappa^2 - 4b_2\kappa^3\right)P_2' - (a_2 - 4b_2\kappa)P_2^{(3)} = 0. \quad (15)$$

Now, differentiating (14) and (15) brings about

$$P_1^{(4)} = \frac{\left(v + 3a_1\kappa^2 - 4b_1\kappa^3\right)P_1''}{a_1 - 4b_2\kappa}, \tag{16}$$

$$P_2^{(4)} = \frac{\left(v + 3a_2\kappa^2 - 4b_2\kappa^3\right)P_2''}{a_2 - 4b_2\kappa}, \tag{17}$$

and then (12) and (13) modify to

$$\left(\frac{3a_1^2\kappa - 15a_1b_1\kappa^2 + b_1\left(v + 20b_1\kappa^3\right)}{a_1 - 4b_1\kappa}\right)P_1'' + \left(b_1\kappa - \omega - a_1\kappa^3\right)P_1 + \frac{p_1P_1}{c_1P_1^2 + d_1P_2^2} + \frac{q_1P_1}{\sqrt{P_1^2 + P_2^2}} + r_1P_1\sqrt{P_1^2 + P_2^2} + \left(\alpha_1P_1^2 + \beta_1P_2^2\right)P_1 = 0, \tag{18}$$

$$\left(\frac{3a_2^2\kappa - 15a_2b_2\kappa^2 + b_2\left(v + 20b_2\kappa^3\right)}{a_2 - 4b_2\kappa}\right)P_2'' + \left(b_2\kappa - \omega - a_2\kappa^3\right)P_2 + \frac{p_2P_2}{c_2P_2^2 + d_2P_1^2} + \frac{q_2P_2}{\sqrt{P_1^2 + P_2^2}} + r_2P_2\sqrt{P_1^2 + P_2^2} + \left(\alpha_2P_2^2 + \beta_2P_1^2\right)P_2 = 0. \tag{19}$$

Next, setting

$$P_2 = \lambda P_1, \tag{20}$$

where $\lambda \neq 0$ and $\lambda \neq 1$, then (18) and (19) can be written as

$$\left(\frac{3a_1^2\kappa - 15a_1b_1\kappa^2 + b_1\left(v + 20b_1\kappa^3\right)}{a_1 - 4b_1\kappa}\right)P_1P_1'' + \frac{p_1}{c_1 + d_1\lambda^2} + \frac{q_1}{\lambda_1}P_1 - \left(\omega + \kappa^3\left(a_1 - b_1\kappa\right)\right)P_1^2 + r_1\lambda_1P_1^3 + \left(\alpha_1 + \beta_1\lambda^2\right)P_1^4 = 0, \tag{21}$$

$$\left(\frac{3a_2^2\kappa - 15a_2b_2\kappa^2 + b_2\left(v + 20b_2\kappa^3\right)}{a_2 - 4b_2\kappa}\right)P_1P_1'' + \frac{p_2}{c_2\lambda^2 + d_2} + \frac{q_2}{\lambda_1}P_1 - \left(\omega + \kappa^3\left(a_2 - b_2\kappa\right)\right)P_1^2 + r_2\lambda_1P_1^3 + \left(\alpha_2\lambda^2 + \beta_2\right)P_1^4 = 0, \tag{22}$$

where $\lambda_1 = \sqrt{1 + \lambda^2}$. Equations (21) and (22) have the same form under the following constraint conditions:

$$\begin{gathered}(a_1 - 4b_1\kappa)\left(3a_2^2\kappa - 15a_2b_2\kappa^2 + b_2\left(v + 20b_2\kappa^3\right)\right) \\ = (a_2 - 4b_2\kappa)\left(3a_1^2\kappa - 15a_1b_1\kappa^2 + b_1\left(v + 20b_1\kappa^3\right)\right) \\ p_1\left(c_2\lambda^2 + d_2\right) = p_2\left(c_1 + d_1\lambda^2\right), \\ q_1 = q_2, \\ a_1 - b_1\kappa = a_2 - b_2\kappa, \\ r_1 = r_2, \\ \alpha_1 + \beta_1\lambda^2 = \alpha_2\lambda^2 + \beta_2.\end{gathered} \tag{23}$$

Therefore Eq. (21) will now be studied, in the subsequent subsection, to reveal cubic–quartic solitons to the model under the conditions given by (23).

2.1.1. Extended trial function. Suppose the formal solution of Eq. (21) is structured in the form [1]:

$$P_1 = \sum_{j=0}^{\varsigma}\varrho_j\psi^j, \tag{24}$$

where

$$(\psi')^2 = \Theta(\psi) = \frac{\Gamma(\psi)}{\Upsilon(\psi)} = \frac{\mu_\sigma\psi^\sigma + \cdots + \mu_1\psi + \mu_0}{\chi_\rho\psi^\rho + \cdots + \chi_1\psi + \chi_0}. \tag{25}$$

Here $\varrho_0, \ldots \varrho_\varsigma$; $\mu_0, \ldots, \mu_\sigma$ and $\chi_0, \ldots, \chi_\rho$ are coefficients that need to be designated, such that the constants ϱ_ς, μ_σ and χ_ρ are non-zero. Next, Eq. (25) is rewritten as

$$\pm(s - s_0) = \int\frac{d\psi}{\sqrt{\Theta(\psi)}} = \int\sqrt{\frac{\Upsilon(\psi)}{\Gamma(\psi)}}d\psi. \tag{26}$$

Balance of the terms P_1P_1'' with P_1^4 in (21) leads to

$$\sigma = \rho + 2\varsigma + 2. \tag{27}$$

For $\rho = 0$, $\varsigma = 1$ and $\sigma = 4$,

$$P_1 = \varrho_0 + \varrho_1\psi. \tag{28}$$

Substituting (28) into (21) yields

$$\begin{gathered}\mu_0 = \mu_0,\ \ \mu_2 = \mu_2,\ \ \mu_4 = \mu_4, \\ \chi_0 = \chi_0,\ \ \varrho_0 = \varrho_0,\ \ \varrho_1 = \varrho_1,\ \ p_1 = 0, \\ v = -\frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4},\ \ \mu_3 = \frac{4\mu_4\left(r_1\lambda_1 + 3\epsilon_2\varrho_0\right)}{3\epsilon_2\varrho_1}, \\ \mu_1 = \frac{4q_1\mu_4 + 2\lambda_1\varrho_0\left(\epsilon_2\left(\mu_2\varrho_1^2 - 4\mu_4\varrho_0^2\right) - 2r_1\lambda_1\mu_4\varrho_0\right)}{\epsilon_2\lambda_1\varrho_1^3}, \\ \omega = \frac{2\mu_4\left(\varrho_0\left(2r_1\lambda_1 + 3\epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4},\end{gathered} \tag{29}$$

where

$$\begin{gathered}\epsilon_1 = 3a_1^2 - 15a_1b_1\kappa + 20b_1^2\kappa, \\ \epsilon_2 = \alpha_1 + \beta_1\lambda^2,\ \ \epsilon_3 = a_1 - 4b_1\kappa.\end{gathered} \tag{30}$$

By the use of these results, (26) is rewritten as below:

$$\pm(s - s_0) = \sqrt{\frac{\chi_0}{\mu_4}} \times \int\frac{d\psi}{\sqrt{\psi^4 + \frac{\mu_3}{\mu_4}\psi^3 + \frac{\mu_2}{\mu_4}\psi^2 + \frac{\mu_1}{\mu_4}\psi + \frac{\mu_0}{\mu_4}}} = \vartheta_1\int\frac{d\psi}{\sqrt{\Theta(\psi)}}. \tag{31}$$

As a consequence, soliton and other solutions to the model are:

For $\Theta(\psi) = (\psi - \delta_1)^4$, $\varrho_0 = -\varrho_1\delta_1$ and $s_0 = 0$, plane wave solutions are recovered as:

$$u(x,t) = \left\{ \pm \frac{\varrho_1 \vartheta_1}{x + \left\{ \frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4} \right\} t} \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{2\mu_4 \left(\varrho_0 \left(2r_1\lambda_1 + 3\epsilon_2\varrho_0 \right) - a_1\kappa^3 + b_1\kappa^4 \right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4} \right) t + \theta \right\} \right], \tag{32}$$

$$v(x,t) = \lambda \left\{ \pm \frac{\varrho_1 \vartheta_1}{x + \left\{ \frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4} \right\} t} \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{2\mu_4 \left(\varrho_0 \left(2r_1\lambda_1 + 3\epsilon_2\varrho_0 \right) - a_1\kappa^3 + b_1\kappa^4 \right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4} \right) t + \theta \right\} \right]. \tag{33}$$

If $\Theta(\psi) = (\psi - \delta_1)^3(\psi - \delta_2)$, $\delta_2 > \delta_1$, $\varrho_0 = -\varrho_1\delta_1$ and $s_0 = 0$, rational function solution is procured as:

$$u(x,t) = \left\{ \frac{4\varrho_1\vartheta_1^2(\delta_2 - \delta_1)}{4\vartheta_1^2 - \left[(\delta_1 - \delta_2)\left(x + \left\{ \frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4} \right\} t \right) \right]^2} \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{2\mu_4 \left(\varrho_0 \left(2r_1\lambda_1 + 3\epsilon_2\varrho_0 \right) - a_1\kappa^3 + b_1\kappa^4 \right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4} \right) t + \theta \right\} \right], \tag{34}$$

$$v(x,t) = \lambda \left\{ \frac{4\varrho_1\vartheta_1^2(\delta_2 - \delta_1)}{4\vartheta_1^2 - \left[(\delta_1 - \delta_2)\left(x + \left\{ \frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4} \right\} t \right) \right]^2} \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{2\mu_4 \left(\varrho_0 \left(2r_1\lambda_1 + 3\epsilon_2\varrho_0 \right) - a_1\kappa^3 + b_1\kappa^4 \right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4} \right) t + \theta \right\} \right]. \tag{35}$$

However, when $\Theta(\psi) = (\psi - \delta_1)^2(\psi - \delta_2)^2$, $\varrho_0 = -\varrho_1\delta_1$ and $s_0 = 0$, cubic–quartic singular solitons are secured as:

$$u(x,t) = \left\{ \frac{\varrho_1(\delta_2 - \delta_1)}{2} \left(1 \mp \coth\left[\frac{\delta_1 - \delta_2}{2\vartheta_1} \left(x + \left\{ \frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4} \right\} t \right) \right] \right) \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{2\mu_4 \left(\varrho_0 \left(2r_1\lambda_1 + 3\epsilon_2\varrho_0 \right) - a_1\kappa^3 + b_1\kappa^4 \right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4} \right) t + \theta \right\} \right], \tag{36}$$

$$v(x,t) = \left\{ \frac{\varrho_1(\delta_2 - \delta_1)}{2} \left(1 \mp \coth\left[\frac{\delta_1 - \delta_2}{2\vartheta_1} \left(x + \left\{ \frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4} \right\} t \right) \right] \right) \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{2\mu_4 \left(\varrho_0 \left(2r_1\lambda_1 + 3\epsilon_2\varrho_0 \right) - a_1\kappa^3 + b_1\kappa^4 \right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4} \right) t + \theta \right\} \right]. \tag{37}$$

Whenever $\Theta(\psi) = (\psi - \delta_1)^2(\psi - \delta_2)(\psi - \delta_3)$, $\delta_1 > \delta_2 > \delta_3$, $\varrho_0 = -\varrho_1\delta_1$ and $s_0 = 0$, cubic–quartic bright soliton is revealed as:

$$u(x,t) = \left\{ \frac{\mathcal{D}_1}{\mathcal{F}_1 + \cosh\left[\mathcal{H}_1 \left(x + \left\{ \frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4} \right\} t \right) \right]} \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{2\mu_4 \left(\varrho_0 \left(2r_1\lambda_1 + 3\epsilon_2\varrho_0 \right) - a_1\kappa^3 + b_1\kappa^4 \right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4} \right) t + \theta \right\} \right], \tag{38}$$

$$v(x,t) = \lambda \left\{ \frac{\mathcal{D}_1}{\mathcal{F}_1 + \cosh\left[\mathcal{H}_1 \left(x + \left\{ \frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4} \right\} t \right) \right]} \right\} \times \exp\left[i \left\{ -\kappa x + \left(\frac{2\mu_4 \left(\varrho_0 \left(2r_1\lambda_1 + 3\epsilon_2\varrho_0 \right) - a_1\kappa^3 + b_1\kappa^4 \right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4} \right) t + \theta \right\} \right], \tag{39}$$

where

$$\mathscr{D}_1 = \frac{2\varrho_1(\delta_1 - \delta_2)(\delta_1 - \delta_3)}{\delta_3 - \delta_2}, \tag{40}$$

$$\mathscr{F}_1 = \frac{2\delta_1 - \delta_2 - \delta_3}{\delta_3 - \delta_2}, \tag{41}$$

$$\mathscr{H}_1 = \frac{\sqrt{(\delta_1 - \delta_2)(\delta_1 - \delta_3)}}{\vartheta_1}. \tag{42}$$

Here, the soliton amplitude and its inverse width are respectively given by $\mathscr{D}_1$ and $\mathscr{H}_1$. The condition $\varrho_1 < 0$ is necessary in order for the solitons that are revealed to exist.

Finally, if $\Theta(\psi) = (\psi - \delta_1)(\psi - \delta_2)(\psi - \delta_3)(\psi - \delta_4)$, $\delta_1 > \delta_2 > \delta_3 > \delta_4$, $\varrho_0 = -\varrho_1\delta_1$ and $s_0 = 0$, Jacobi elliptic function (JEF) solutions are derived as:

$$u(x,t) = \left\{\frac{\mathscr{D}_2}{\mathscr{F}_2 + \mathrm{sn}^2\left[\mathscr{H}_j\left(x + \left\{\frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4}\right\}t\right), k\right]}\right\}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(\varrho_0\left(2r_1\lambda_1 + 3\epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4}\right)t + \theta\right\}\right], \tag{43}$$

$$v(x,t) = \lambda\left\{\frac{\mathscr{D}_2}{\mathscr{F}_2 + \mathrm{sn}^2\left[\mathscr{H}_j\left(x + \left\{\frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4}\right\}t\right), k\right]}\right\}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(\varrho_0\left(2r_1\lambda_1 + 3\epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4}\right)t + \theta\right\}\right], \tag{44}$$

where

$$k = \frac{(\delta_2 - \delta_3)(\delta_1 - \delta_4)}{(\delta_1 - \delta_3)(\delta_2 - \delta_4)}, \tag{45}$$

$$\mathscr{D}_2 = \frac{\varrho_1(\delta_1 - \delta_2)(\delta_4 - \delta_2)}{\delta_1 - \delta_4}, \tag{46}$$

$$\mathscr{F}_2 = \frac{\delta_4 - \delta_2}{\delta_1 - \delta_4}, \tag{47}$$

$$\mathscr{H}_j = \frac{(-1)^j\sqrt{(\delta_1 - \delta_3)(\delta_2 - \delta_4)}}{2\vartheta_1} \text{ for } j = 2,3. \tag{48}$$

Here, δ_j for $j = 1, 4$ are the zeros of $\Theta(\psi) = 0$.

Remark 1. When the modulus $k \to 1$, from (43) and (44), cubic–quartic singular solitons fall out

$$u(x,t) = \left\{\frac{\mathscr{D}_2}{\mathscr{F}_2 + \tanh^2\left[\mathscr{H}_j\left(x + \left\{\frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4}\right\}t\right)\right]}\right\}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(\varrho_0\left(2r_1\lambda_1 + 3\epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4}\right)t + \theta\right\}\right], \tag{49}$$

$$v(x,t) = \lambda\left\{\frac{\mathscr{D}_2}{\mathscr{F}_2 + \tanh^2\left[\mathscr{H}_j\left(x + \left\{\frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4}\right\}t\right)\right]}\right\}$$
$$\times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(\varrho_0\left(2r_1\lambda_1 + 3\epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4}\right)t + \theta\right\}\right], \tag{50}$$

where $\delta_3 = \delta_4$.

Remark 2. If $k \to 0$, in this case, periodic singular solutions are

$$u(x,t) = \left\{ \frac{\mathcal{D}_2}{\mathcal{F}_2 + \sin^2\left[\mathcal{H}_j\left(x + \left\{\frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4}\right\}t\right)\right]} \right\} \times \exp\left[i\left\{ -\kappa x + \left(\frac{2\mu_4\left(\varrho_0\left(2r_1\lambda_1 + 3\epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4} \right)t + \theta \right\}\right], \tag{51}$$

$$v(x,t) = \lambda\left\{ \frac{\mathcal{D}_2}{\mathcal{F}_2 + \sin^2\left[\mathcal{H}_j\left(x + \left\{\frac{2\kappa\epsilon_1\mu_4 + \epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4}\right\}t\right)\right]} \right\} \times \exp\left[i\left\{ -\kappa x + \left(\frac{2\mu_4\left(\varrho_0\left(2r_1\lambda_1 + 3\epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) - \epsilon_2\mu_2\varrho_1^2}{2\mu_4} \right)t + \theta \right\}\right], \tag{52}$$

where $\delta_2 = \delta_3$.

2.2. Case 2: (n = 2)

Upon substituting (8) and (9) into (4) and (5), the resulting real parts are

$$3a_1\kappa P_1'' + b_1\left(P_1^{(4)} - 6\kappa^2 P_1''\right) + \frac{p_1P_1}{c_1P_1^4 + d_1P_1^2P_2^2 + e_1P_2^4} + \frac{q_1P_1}{f_1P_1^2 + g_1P_2^2} + \left(b_1\kappa^4 - \omega - a_1\kappa^3\right)P_1 + \alpha_1P_1^3 + \beta_1P_1P_2^2 + \eta_1P_1^3P_2^2 + \zeta_1P_1P_2^4 + \xi_1P_1^5 = 0, \tag{53}$$

$$3a_2\kappa P_2'' + b_2\left(P_2^{(4)} - 6\kappa^2 P_2''\right) + \frac{p_2P_2}{c_2P_2^4 + d_2P_2^2P_1^2 + e_2P_1^4} + \frac{q_2P_2}{f_2P_2^2 + g_2P_1^2} + \left(b_2\kappa^4 - \omega - a_2\kappa^3\right)P_2 + \alpha_2P_2^3 + \beta_2P_2P_1^2 + \eta_2P_2^3P_1^2 + \zeta_2P_2P_1^4 + \xi_2P_2^5 = 0, \tag{54}$$

and the imaginary parts are given by (14) and (15) and in this case, (16) and (17) are also satisfied. By virtue of (16) and (17), Eqs. (53) and (54) can be written as

$$\left(\frac{3a_1^2\kappa - 15a_1b_1\kappa^2 + b_1\left(v + 20b_1\kappa^3\right)}{a_1 - 4b_1\kappa}\right)P_1'' + \frac{p_1P_1}{c_1P_1^4 + d_1P_1^2P_2^2 + e_1P_2^4} + \frac{q_1P_1}{f_1P_1^2 + g_1P_2^2} + \left(b_1\kappa^4 - \omega - a_1\kappa^3\right)P_1 + \alpha_1P_1^3 + \beta_1P_1P_2^2 + \eta_1P_1^3P_2^2 + \zeta_1P_1P_2^4 + \xi_1P_1^5 = 0, \tag{55}$$

$$\left(\frac{3a_2^2\kappa - 15a_2b_2\kappa^2 + b_2\left(v + 20b_2\kappa^3\right)}{a_2 - 4b_2\kappa}\right)P_2'' + \frac{p_2P_2}{c_2P_2^4 + d_2P_2^2P_1^2 + e_2P_1^4} + \frac{q_2P_2}{f_2P_2^2 + g_2P_1^2} + \left(b_2\kappa^4 - \omega - a_2\kappa^3\right)P_2 + \alpha_2P_2^3 + \beta_2P_2P_1^2 + \eta_2P_2^3P_1^2 + \zeta_2P_2P_1^4 + \xi_2P_2^5 = 0. \tag{56}$$

Next, setting

$$P_2 = \lambda P_1, \tag{57}$$

where $\lambda \neq 0$ and $\lambda \neq 1$ then, (55) and (56) become

$$\left(\frac{3a_1^2\kappa - 15a_1b_1\kappa^2 + b_1\left(v + 20b_1\kappa^3\right)}{a_1 - 4b_1\kappa}\right)P_1^3P_1'' + \frac{p_1}{c_1 + d_1\lambda^2 + e_1\lambda^4} + \left(\frac{q_1}{f_1 + g_1\lambda^2}\right)P_1^2 - \left(\omega + \kappa^3\left(a_1 - b_1\kappa\right)\right)P_1^4 + \left(\alpha_1 + \beta_1\lambda^2\right)P_1^6 + \left(\zeta_1\lambda^4 + \eta_1\lambda^2 + \xi_1\right)P_1^8 = 0, \tag{58}$$

$$\left(\frac{3a_2^2\kappa - 15a_2b_2\kappa^2 + b_2\left(v + 20b_2\kappa^3\right)}{a_2 - 4b_2\kappa}\right)P_1^3P_1'' + \frac{p_2}{c_2\lambda^4 + d_2\lambda^2 + e_2} + \left(\frac{q_2}{f_2\lambda^2 + g_2}\right)P_1^2 - \left(\omega + \kappa^3\left(a_2 - b_2\kappa\right)\right)P_1^4 + \left(\alpha_2\lambda^2 + \beta_2\right)P_1^6 + \left(\zeta_2 + \eta_2\lambda^2 + \xi_2\lambda^4\right)P_1^8 = 0. \tag{59}$$

For extracting closed form solutions, the following transformation is applied to Eqs. (58) and (59):

$$P_1 = U^{\frac{1}{2}}. \tag{60}$$

Then Eqs. (58) and (59) change to

$$\left(\frac{3a_1^2\kappa - 15a_1b_1\kappa^2 + b_1\left(v + 20b_1\kappa^3\right)}{4(a_1 - 4b_1\kappa)}\right)\left(2UU'' - (U')^2\right) + \frac{p_1}{c_1 + d_1\lambda^2 + e_1\lambda^4} + \left(\frac{q_1}{f_1 + g_1\lambda^2}\right)U \\ - \left(\omega + \kappa^3\left(a_1 - b_1\kappa\right)\right)U^2 + \left(\alpha_1 + \beta_1\lambda^2\right)U^3 + \left(\zeta_1\lambda^4 + \eta_1\lambda^2 + \xi_1\right)U^4 = 0, \tag{61}$$

$$\left(\frac{3a_2^2\kappa - 15a_2b_2\kappa^2 + b_2\left(v + 20b_2\kappa^3\right)}{4(a_2 - 4b_2\kappa)}\right)\left(2UU'' - (U')^2\right) + \frac{p_2}{c_2\lambda^4 + d_2\lambda^2 + e_2} + \left(\frac{q_2}{f_2\lambda^2 + g_2}\right)U \\ - \left(\omega + \kappa^3\left(a_2 - b_1\kappa\right)\right)U^2 + \left(\alpha_2\lambda^2 + \beta_2\right)U^3 + \left(\zeta_2 + \eta_2\lambda^2 + \xi_2\lambda^4\right)U^4 = 0. \tag{62}$$

Equations (61) and (62) have the same form under the constraint conditions given by

$$\begin{gathered}(a_1 - 4b_1\kappa)\left(3a_2^2\kappa - 15a_2b_2\kappa^2 + b_2\left(v + 20b_2\kappa^3\right)\right) \\ = (a_2 - 4b_2\kappa)\left(3a_1^2\kappa - 15a_1b_1\kappa^2 + b_1\left(v + 20b_1\kappa^3\right)\right), \\ p_1\left(c_2\lambda^4 + d_2\lambda^2 + e_2\right) = p_2\left(c_1 + d_1\lambda^2 + e_1\lambda^4\right), \\ q_1(f_2\lambda^2 + g_2) = q_2(f_1 + g_1\lambda^2), \\ a_1 - b_1\kappa = a_2 - b_2\kappa, \\ \alpha_1 + \beta_1\lambda^2 = \alpha_2\lambda^2 + \beta_2, \\ \zeta_1\lambda^4 + \eta_1\lambda^2 + \xi_1 = \zeta_2 + \eta_2\lambda^2 + \xi_2\lambda^4.\end{gathered} \tag{63}$$

So Eq. (61) will now be examined, in the next subsection, in order to secure cubic–quartic solitons to the governing equation considering the conditions (63).

2.2.1. Extended trial function. Balance of the terms UU'' or $(U')^2$ with U^4 appeared in (61) gives rise to

$$\sigma = \rho + 2\varsigma + 2. \tag{64}$$

For $\rho = 0$, $\varsigma = 1$ and $\sigma = 4$,

$$U = \varrho_0 + \varrho_1\psi. \tag{65}$$

Inserting (65) into (61) gives

$$\begin{gathered}\mu_0 = \mu_0, \quad \mu_1 = \mu_1, \quad \mu_4 = \mu_4, \\ \chi_0 = \chi_0, \quad \varrho_0 = \varrho_0, \quad \varrho_1 = \varrho_1, \quad q_1 = 0, \\ v = -\frac{3\kappa\epsilon_1\mu_4 + 4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}, \quad \mu_3 = \frac{\mu_4\left(3\alpha_1 + 3\beta_1\lambda^2 + 8\epsilon_2\varrho_0\right)}{2\epsilon_2\varrho_1}, \\ \mu_2 = \frac{\epsilon_4\left(3\mu_4\varrho_0^3\left(\alpha_1 + \beta_1\lambda^2 + 2\epsilon_2\varrho_0\right) + 2\epsilon_2\mu_1\varrho_0\varrho_1^3 - 2\epsilon_2\mu_0\varrho_1^4\right) - 6p_1\mu_4}{2\epsilon_2\epsilon_4\varrho_0^2\varrho_1^2}, \\ \omega = \frac{3p_1\mu_4 + \epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1 + \beta_1\lambda^2 + 2\epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) + \epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2},\end{gathered} \tag{66}$$

where

$$\begin{gathered}\epsilon_1 = 3a_1^2 - 15a_1b_1\kappa + 20b_1^2\kappa^2, \quad \epsilon_2 = \zeta_1\lambda^4 + \eta_1\lambda^2 + \xi_1, \quad \epsilon_3 = a_1 - 4b_1\kappa, \\ \epsilon_4 = c_1 + d_1\lambda^2 + e_1\lambda^4, \quad \epsilon_5 = \mu_0\varrho_1 - \mu_1\varrho_0.\end{gathered} \tag{67}$$

Then, (26) shapes up

$$\pm(s - s_0) = \sqrt{\frac{\chi_0}{\mu_4}}\int\frac{d\psi}{\sqrt{\psi^4 + \frac{\mu_3}{\mu_4}\psi^3 + \frac{\mu_2}{\mu_4}\psi^2 + \frac{\mu_1}{\mu_4}\psi + \frac{\mu_0}{\mu_4}}} = \vartheta_2\int\frac{d\psi}{\sqrt{\Theta(\psi)}}. \tag{68}$$

Integrating the last equation, one recovers the following exact solutions to the model:

For $\Theta(\psi) = (\psi - \delta_1)^4$, $\varrho_0 = -\varrho_1\delta_1$ and $s_0 = 0$, plane wave solutions are:

$$u(x,t) = \left\{\pm\frac{\varrho_1\vartheta_2}{x + \left\{\frac{2\kappa\epsilon_1\mu_4 + 4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t}\right\}^{1/2} \\ \times \exp\left[i\left\{-\kappa x + \left(\frac{3p_1\mu_4 + \epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1 + \beta_1\lambda^2 + \epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) + \epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t + \theta\right\}\right], \tag{69}$$

$$v(x,t)=\lambda\left\{\pm\frac{\varrho_1\vartheta_2}{x+\left\{\frac{2\kappa\epsilon_1\mu_4+4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t}\right\}^{1/2}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{3p_1\mu_4+\epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1+\beta_1\lambda^2+\epsilon_2\varrho_0\right)-a_1\kappa^3+b_1\kappa^4\right)+\epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t+\theta\right\}\right]. \tag{70}$$

If $\Theta(\psi)=(\psi-\delta_1)^3(\psi-\delta_2)$, $\delta_2>\delta_1$, $\varrho_0=-\varrho_1\delta_1$ and $s_0=0$, rational function solution is:

$$u(x,t)=\left\{\frac{4\varrho_1\vartheta_2^2(\delta_2-\delta_1)}{4\vartheta_2^2-\left[(\delta_1-\delta_2)\left(x+\left\{\frac{3\kappa\epsilon_1\mu_4+4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t\right)\right]^2}\right\}^{1/2}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{3p_1\mu_4+\epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1+\beta_1\lambda^2+\epsilon_2\varrho_0\right)-a_1\kappa^3+b_1\kappa^4\right)+\epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t+\theta\right\}\right], \tag{71}$$

$$v(x,t)=\lambda\left\{\frac{4\varrho_1\vartheta_2^2(\delta_2-\delta_1)}{4\vartheta_2^2-\left[(\delta_1-\delta_2)\left(x+\left\{\frac{3\kappa\epsilon_1\mu_4+4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t\right)\right]^2}\right\}^{1/2}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{3p_1\mu_4+\epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1+\beta_1\lambda^2+\epsilon_2\varrho_0\right)-a_1\kappa^3+b_1\kappa^4\right)+\epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t+\theta\right\}\right]. \tag{72}$$

However, when $\Theta(\psi)=(\psi-\delta_1)^2(\psi-\delta_2)^2$, $\varrho_0=-\varrho_1\delta_1$ and $s_0=0$, cubic–quartic singular solitons are:

$$u(x,t)=\left\{\frac{\varrho_1(\delta_2-\delta_1)}{2}\left(1\mp\coth\left[\frac{\delta_1-\delta_2}{2\vartheta_2}\left(x+\left\{\frac{3\kappa\epsilon_1\mu_4+4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4}\right\}t\right)\right]\right)\right\}^{1/2}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{3p_1\mu_4+\epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1+\beta_1\lambda^2+\epsilon_2\varrho_0\right)-a_1\kappa^3+b_1\kappa^4\right)+\epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t+\theta\right\}\right], \tag{73}$$

$$v(x,t)=\lambda\left\{\frac{\varrho_1(\delta_2-\delta_1)}{2}\left(1\mp\coth\left[\frac{\delta_1-\delta_2}{2\vartheta_2}\left(x+\left\{\frac{3\kappa\epsilon_1\mu_4+4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{2b_1\mu_4}\right\}t\right)\right]\right)\right\}^{1/2}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{3p_1\mu_4+\epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1+\beta_1\lambda^2+\epsilon_2\varrho_0\right)-a_1\kappa^3+b_1\kappa^4\right)+\epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t+\theta\right\}\right]. \tag{74}$$

Whenever $\Theta(\psi)=(\psi-\delta_1)^2(\psi-\delta_2)(\psi-\delta_3)$, $\delta_1>\delta_2>\delta_3$, $\varrho_0=-\varrho_1\delta_1$ and $s_0=0$, cubic–quartic bright soliton is:

$$u(x,t)=\left\{\frac{\mathcal{D}_3}{\mathcal{F}_3+\cosh\left[\mathcal{H}_4\left(x+\left\{\frac{3\kappa\epsilon_1\mu_4+4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t\right)\right]}\right\}^{1/2}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{3p_1\mu_4+\epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1+\beta_1\lambda^2+\epsilon_2\varrho_0\right)-a_1\kappa^3+b_1\kappa^4\right)+\epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t+\theta\right\}\right], \tag{75}$$

$$v(x,t)=\lambda\left\{\frac{\mathcal{D}_3}{\mathcal{F}_3+\cosh\left[\mathcal{H}_4\left(x+\left\{\frac{3\kappa\epsilon_1\mu_4+4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t\right)\right]}\right\}^{1/2}$$
$$\times\exp\left[i\left\{-\kappa x+\left(\frac{3p_1\mu_4+\epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1+\beta_1\lambda^2+\epsilon_2\varrho_0\right)-a_1\kappa^3+b_1\kappa^4\right)+\epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t+\theta\right\}\right], \tag{76}$$

where

$$\mathcal{D}_3=\frac{2\varrho_1(\delta_1-\delta_2)(\delta_1-\delta_3)}{\delta_3-\delta_2}, \tag{77}$$

$$\mathcal{F}_3 = \frac{2\delta_1 - \delta_2 - \delta_3}{\delta_3 - \delta_2}, \tag{78}$$

$$\mathcal{H}_4 = \frac{\sqrt{(\delta_1 - \delta_2)(\delta_1 - \delta_3)}}{\vartheta_2}. \tag{79}$$

Here, the soliton amplitude and its inverse width are respectively indicated by $\mathcal{D}_3$ and $\mathcal{H}_4$. The condition $\varrho_1 < 0$ is necessary in order for the solitons obtained to exist.

Finally, if $\Theta(\psi) = (\psi - \delta_1)(\psi - \delta_2)(\psi - \delta_3)(\psi - \delta_4)$, $\delta_1 > \delta_2 > \delta_3 > \delta_4$, $\varrho_0 = -\varrho_1\delta_2$ and $s_0 = 0$, JEF solutions are:

$$u(x,t) = \left\{\frac{\mathcal{D}_4}{\mathcal{F}_4 + \mathrm{sn}^2\left[\mathcal{H}_j\left(x + \left\{\frac{3\kappa\epsilon_1\mu_4 + 4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t\right), k\right]}\right\}^{1/2} \times \exp\left[i\left\{-\kappa x + \left(\frac{3p_1\mu_4 + \epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1 + \beta_1\lambda^2 + \epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) + \epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t + \theta\right\}\right], \tag{80}$$

$$v(x,t) = \lambda\left\{\frac{\mathcal{D}_4}{\mathcal{F}_4 + \mathrm{sn}^2\left[\mathcal{H}_j\left(x + \left\{\frac{3\kappa\epsilon_1\mu_4 + 4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t\right), k\right]}\right\}^{1/2} \times \exp\left[i\left\{-\kappa x + \left(\frac{3p_1\mu_4 + \epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1 + \beta_1\lambda^2 + \epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) + \epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t + \theta\right\}\right], \tag{81}$$

where

$$k = \frac{(\delta_2 - \delta_3)(\delta_1 - \delta_4)}{(\delta_1 - \delta_3)(\delta_2 - \delta_4)}, \tag{82}$$

$$\mathcal{D}_4 = \frac{\varrho_1(\delta_1 - \delta_2)(\delta_4 - \delta_2)}{\delta_1 - \delta_4}, \tag{83}$$

$$\mathcal{F}_4 = \frac{\delta_4 - \delta_2}{\delta_1 - \delta_4}, \tag{84}$$

$$\mathcal{H}_j = \frac{(-1)^j\sqrt{(\delta_1 - \delta_3)(\delta_2 - \delta_4)}}{2\vartheta_2} \text{ for } j = 5,6. \tag{85}$$

Remark 3. For $k \to 1$, from (80) and (81), cubic–quartic singular optical solitons are constructed as

$$u(x,t) = \left\{\frac{\mathcal{D}_4}{\mathcal{F}_4 + \tanh^2\left[\mathcal{H}_j\left(x + \left\{\frac{3\kappa\epsilon_1\mu_4 + 4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t\right)\right]}\right\}^{1/2} \times \exp\left[i\left\{-\kappa x + \left(\frac{3p_1\mu_4 + \epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1 + \beta_1\lambda^2 + \epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) + \epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t + \theta\right\}\right], \tag{86}$$

$$v(x,t) = \lambda\left\{\frac{\mathcal{D}_4}{\mathcal{F}_4 + \tanh^2\left[\mathcal{H}_j\left(x + \left\{\frac{3\kappa\epsilon_1\mu_4 + 4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t\right)\right]}\right\}^{1/2} \times \exp\left[i\left\{-\kappa x + \left(\frac{3p_1\mu_4 + \epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1 + \beta_1\lambda^2 + \epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) + \epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t + \theta\right\}\right], \tag{87}$$

where $\delta_3 = \delta_4$.

Remark 4. If $k \to 0$, in this case, periodic singular solutions are

$$u(x,t) = \left\{ \frac{\mathcal{D}_4}{\mathcal{F}_4 + \sin^2\left[\mathcal{H}_j\left(x + \left\{\frac{3\kappa\epsilon_1\mu_4 + 4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t\right)\right]} \right\}^{1/2} \times \exp\left[i\left\{-\kappa x + \left(\frac{3p_1\mu_4 + \epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1 + \beta_1\lambda^2 + \epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) + \epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t + \theta\right\}\right], \tag{88}$$

$$v(x,t) = \lambda\left\{ \frac{\mathcal{D}_4}{\mathcal{F}_4 + \sin^2\left[\mathcal{H}_j\left(x + \left\{\frac{3\kappa\epsilon_1\mu_4 + 4\epsilon_2\epsilon_3\varrho_1^2\chi_0}{3b_1\mu_4}\right\}t\right)\right]} \right\}^{1/2} \times \exp\left[i\left\{-\kappa x + \left(\frac{3p_1\mu_4 + \epsilon_4\left(3\mu_4\varrho_0^2\left(\varrho_0\left(\alpha_1 + \beta_1\lambda^2 + \epsilon_2\varrho_0\right) - a_1\kappa^3 + b_1\kappa^4\right) + \epsilon_2\epsilon_5\varrho_1^3\right)}{3\epsilon_4\mu_4\varrho_0^2}\right)t + \theta\right\}\right], \tag{89}$$

where $\delta_2 = \delta_3$.

2.3. Case 3: (n = 3)

Upon inserting (8) and (9) into (6) and (7), the real part equations are

$$3a_1\kappa P_1'' + b_1\left(P_1^{(4)} - 6\kappa^2 P_1''\right) + \frac{p_1P_1}{c_1P_1^6 + d_1P_1^4P_2^2 + e_1P_1^2P_2^4 + f_1P_2^6} + \frac{q_1P_1}{\sqrt{P_1^2 + P_2^2}\left(g_1P_1^2 + h_1P_2^2\right)} + \left(b_1\kappa^4 - \omega - a_1\kappa^3\right)P_1 + \alpha_1P_1^3\sqrt{P_1^2 + P_2^2} + \beta_1P_1P_2^2\sqrt{P_1^2 + P_2^2} + \eta_1P_1^5P_2^2 + \zeta_1P_1^3P_2^4 + \theta_1P_1P_2^6 + \xi_1P_1^7 = 0, \tag{90}$$

$$3a_2\kappa P_2'' + b_2\left(P_2^{(4)} - 6\kappa^2 P_2''\right) + \frac{p_2P_2}{c_2P_2^6 + d_2P_2^4P_1^2 + e_2P_2^2P_1^4 + f_2P_1^6} + \frac{q_2P_2}{\sqrt{P_2^2 + P_1^2}\left(g_2P_2^2 + h_2P_1^2\right)} + \left(b_2\kappa^4 - \omega - a_2\kappa^3\right)P_2 + \alpha_2P_2^3\sqrt{P_2^2 + P_1^2} + \beta_2P_2P_1^2\sqrt{P_2^2 + P_1^2} + \eta_2P_2^5P_1^2 + \zeta_2P_2^3P_1^4 + \theta_2P_2P_1^6 + \xi_2P_2^7 = 0, \tag{91}$$

and the imaginary parts are given by (14) and (15) and in this case, (16) and (17) are also satisfied. By the help of (16) and (17), Eqs. (90) and (91) modify to

$$\left(\frac{3a_1^2\kappa - 15a_1b_1\kappa^2 + b_1\left(v + 20b_1\kappa^3\right)}{a_1 - 4b_1\kappa}\right)P_1'' + \frac{p_1P_1}{c_1P_1^6 + d_1P_1^4P_2^2 + e_1P_1^2P_2^4 + f_1P_2^6} + \frac{q_1P_1}{\sqrt{P_1^2 + P_2^2}\left(g_1P_1^2 + h_1P_2^2\right)} + \left(b_1\kappa^4 - \omega - a_1\kappa^3\right)P_1 + \alpha_1P_1^3\sqrt{P_1^2 + P_2^2} + \beta_1P_1P_2^2\sqrt{P_1^2 + P_2^2} + \eta_1P_1^5P_2^2 + \zeta_1P_1^3P_2^4 + \theta_1P_1P_2^6 + \xi_1P_1^7 = 0, \tag{92}$$

$$\left(\frac{3a_2^2\kappa - 15a_2b_2\kappa^2 + b_2\left(v + 20b_2\kappa^3\right)}{a_2 - 4b_2\kappa}\right)P_2'' + \frac{p_2P_2}{c_2P_2^6 + d_2P_2^4P_1^2 + e_2P_2^2P_1^4 + f_2P_1^6} + \frac{q_2P_2}{\sqrt{P_2^2 + P_1^2}\left(g_2P_2^2 + h_2P_1^2\right)} + \left(b_2\kappa^4 - \omega - a_2\kappa^3\right)P_2 + \alpha_2P_2^3\sqrt{P_2^2 + P_1^2} + \beta_2P_2P_1^2\sqrt{P_2^2 + P_1^2} + \eta_2P_2^5P_1^2 + \zeta_2P_2^3P_1^4 + \theta_2P_2P_1^6 + \xi_2P_2^7 = 0. \tag{93}$$

Next, setting

$$P_2 = \lambda P_1, \tag{94}$$

where $\lambda \neq 0$ and $\lambda \neq 1$, then (92) and (93) become

$$\left(\frac{3a_1^2\kappa - 15a_1b_1\kappa^2 + b_1\left(v + 20b_1\kappa^3\right)}{a_1 - 4b_1\kappa}\right)P_1^5P_1'' + \frac{p_1}{c_1 + d_1\lambda^2 + e_1\lambda^4 + f_1\lambda^6} + \frac{q_1P_1^3}{\lambda_1\left(g_1 + h_1\lambda^2\right)} - \left(\omega + \kappa^3\left(a_1 - b_1\kappa\right)\right)P_1^6 + \lambda_1\left(\alpha_1 + \beta_1\lambda^2\right)P_1^9 + \left(\zeta_1\lambda^4 + \eta_1\lambda^2 + \theta_1\lambda^6 + \xi_1\right)P_1^{12} = 0, \tag{95}$$

$$\left(\frac{3a_2^2\kappa - 15a_2b_2\kappa^2 + b_2\left(v + 20b_2\kappa^3\right)}{a_2 - 4b_2\kappa}\right)P_1^5P_1'' + \frac{p_2}{c_2\lambda^6 + d_2\lambda^4 + e_2\lambda^2 + f_2} + \frac{q_2P_1^3}{\lambda_1\left(g_2\lambda^2 + h_2\right)} - \left(\omega + \kappa^3\left(a_2 - b_2\kappa\right)\right)P_1^6 + \lambda_1\left(\alpha_2\lambda^2 + \beta_2\right)P_1^9 + \left(\zeta_2\lambda^2 + \eta_2\lambda^4 + \theta_2 + \xi_2\lambda^6\right)P_1^{12} = 0, \tag{96}$$

where $\lambda_1 = \sqrt{1+\lambda^2}$. For recovering closed form solutions, the transformation

$$P_1 = U^{\frac{1}{3}}, \tag{97}$$

is applied to Eqs. (95) and (96). Thus Eqs. (95) and (96) change to

$$\begin{gathered}\left(\frac{3a_1^2\kappa - 15a_1b_1\kappa^2 + b_1\left(v + 20b_1\kappa^3\right)}{9(a_1 - 4b_1\kappa)}\right)\left(3UU'' - 2(U')^2\right) + \frac{p_1}{c_1 + d_1\lambda^2 + e_1\lambda^4 + f_1\lambda^4} + \frac{q_1U}{\lambda_1\left(g_1 + h_1\lambda^2\right)} \\ - \left(\omega + \kappa^3\left(a_1 - b_1\kappa\right)\right)U^2 + \lambda_1\left(\alpha_1 + \beta_1\lambda^2\right)U^3 + \left(\zeta_1\lambda^4 + \eta_1\lambda^2 + \theta_1\lambda^6 + \xi_1\right)U^4 = 0,\end{gathered} \tag{98}$$

$$\begin{gathered}\left(\frac{3a_2^2\kappa - 15a_2b_2\kappa^2 + b_2\left(v + 20b_2\kappa^3\right)}{9(a_2 - 4b_2\kappa)}\right)\left(3UU'' - 2(U')^2\right) + \frac{p_2}{c_2\lambda^6 + d_2\lambda^4 + e_1\lambda^2 + f_2} + \frac{q_2U}{\lambda_1\left(g_2\lambda^2 + h_2\right)} \\ - \left(\omega + \kappa^3\left(a_2 - b_2\kappa\right)\right)U^2 + \lambda_1\left(\alpha_2\lambda^2 + \beta_2\right)U^3 + \left(\zeta_2\lambda^2 + \eta_2\lambda^4 + \theta_2 + \xi_2\lambda^6\right)U^4 = 0.\end{gathered} \tag{99}$$

Equations (98) and (99) have the same form under the constraints

$$\begin{gathered}(a_1 - 4b_1\kappa)\left(3a_2^2\kappa - 15a_2b_2\kappa^2 + b_2\left(v + 20b_2\kappa^3\right)\right) = (a_2 - 4b_2\kappa)\left(3a_1^2\kappa - 15a_1b_1\kappa^2 + b_1\left(v + 20b_1\kappa^3\right)\right), \\ p_1\left(c_2\lambda^6 + d_2\lambda^4 + e_2\lambda^2 + f_2\right) = p_2\left(c_1 + d_1\lambda^2 + e_1\lambda^4 + f_1\lambda^6\right), \\ q_1\left(g_2\lambda^2 + h_2\right) = q_2\left(g_1 + h_1\lambda^2\right) \\ a_1 - b_1\kappa = a_2 - b_2\kappa, \\ \alpha_1 + \beta_1\lambda^2 = \alpha_2\lambda^2 + \beta_2, \\ \zeta_1\lambda^4 + \eta_1\lambda^2 + \theta_1\lambda^6 + \xi_1 = \zeta_2\lambda^2 + \eta_2\lambda^4 + \theta_2 + \xi_2\lambda^6.\end{gathered} \tag{100}$$

Therefore Eq. (98) will now be investigated, in the following subsection, in order procuring cubic–quartic solitons to the considered model under the conditions (100).

2.3.1. Extended trial function. Balance of the terms UU'' or $(U')^2$ with U^4 in (98) implies

$$\sigma = \rho + 2\varsigma + 2. \tag{101}$$

For $p = 0$, $\varsigma = 1$ and $\sigma = 4$,

$$U = \varrho_0 + \varrho_1\psi. \tag{102}$$

Plugging (102) into (98) leads to

$$\begin{gathered}\mu_0 = \mu_0, \quad \mu_2 = \mu_2, \quad \mu_4 = \mu_4, \\ \chi_0 = \chi_0, \quad \varrho_0 = \varrho_0, \quad \varrho_1 = \varrho_1, \\ p_1 = -\frac{\epsilon_4\left(40q_1\mu_4\varrho_0 + \epsilon_5\lambda_1\left(5\epsilon_2\left(5\mu_4\varrho_0^4 - \mu_2\varrho_0^2\varrho_1^2 + \mu_0\varrho_1^4\right) + 16\epsilon_6\lambda_1\mu_4\varrho_0^3\right)\right)}{10\epsilon_5\lambda_1\mu_4}, \\ v = -\frac{4\kappa\epsilon_1\mu_4 + 9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}, \quad \mu_3 = \frac{4\mu_4\left(2\epsilon_6\lambda_1 + 5\epsilon_2\varrho_0\right)}{5\epsilon_2\varrho_1}, \\ \mu_1 = \frac{2\epsilon_5\lambda_1\varrho_0\left(5\epsilon_2\mu_2\varrho_1^2 - 4\mu_4\varrho_0\left(3\epsilon_6\lambda_1 + 5\epsilon_2\varrho_0\right)\right) - 40q_1\mu_4}{5\epsilon_2\epsilon_5\lambda_1\varrho_1^3}, \\ \omega = \frac{2\mu_4\left(10\kappa^3\left(b_1\kappa - a_1\right) + 3\varrho_0\left(4\epsilon_6\lambda_1 + 5\epsilon_2\varrho_0\right)\right) - 5\epsilon_2\mu_2\varrho_1^2}{20\mu_4},\end{gathered} \tag{103}$$

where

$$\begin{gathered}\epsilon_1 = 3a_1^2 - 15a_1b_1\kappa + 20b_1^2\kappa^2, \\ \epsilon_2 = \zeta_1\lambda^4 + \eta_1\lambda^2 + \theta_1\lambda^6 + \xi_1, \\ \epsilon_3 = a_1 - 4b_1\kappa, \quad \epsilon_4 = c_1 + d_1\lambda^2 + e_1\lambda^4 + f_1\lambda^6, \\ \epsilon_5 = g_1 + h_1\lambda^2, \quad \epsilon_6 = \alpha_1 + \beta_1\lambda^2.\end{gathered} \tag{104}$$

Employing these results, one can be rewritten (26) as

$$\pm(s-s_0)=\sqrt{\frac{\chi_0}{\mu_4}}\int\frac{d\psi}{\sqrt{\psi^4+\frac{\mu_3}{\mu_4}\psi^3+\frac{\mu_2}{\mu_4}\psi^2+\frac{\mu_1}{\mu_4}\psi+\frac{\mu_0}{\mu_4}}}=\vartheta_3\int\frac{d\psi}{\sqrt{\Theta(\psi)}}. \tag{105}$$

As a results, the solutions for the governing model are listed as follows:

For $\Theta(\psi)=(\psi-\delta_1)^4$, $\varrho_0=-\varrho_1\delta_1$ and $s_0=0$, plane wave solutions are:

$$\begin{aligned}u(x,t)&=\left\{\pm\frac{\varrho_1\vartheta_3}{x+\left\{\frac{4\kappa\epsilon_1\mu_4+9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t}\right\}^{1/3}\\&\times\exp\left[i\left\{-\kappa x+\left(\frac{2\mu_4\left(10\kappa^3\left(b_1\kappa-a_1\right)+3\varrho_0\left(4\epsilon_6\lambda_1+5\epsilon_2\varrho_0\right)\right)-5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t+\theta\right\}\right],\end{aligned} \tag{106}$$

$$\begin{aligned}v(x,t)&=\lambda\left\{\pm\frac{\varrho_1\vartheta_3}{x+\left\{\frac{4\kappa\epsilon_1\mu_4+9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t}\right\}^{1/3}\\&\times\exp\left[i\left\{-\kappa x+\left(\frac{2\mu_4\left(10\kappa^3\left(b_1\kappa-a_1\right)+3\varrho_0\left(4\epsilon_6\lambda_1+5\epsilon_2\varrho_0\right)\right)-5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t+\theta\right\}\right].\end{aligned} \tag{107}$$

If $\Theta(\psi)=(\psi-\delta_1)^3(\psi-\delta_2)$, $\delta_2>\delta_1$, $\varrho_0=-\varrho_1\delta_1$ and $s_0=0$, rational function solution is:

$$\begin{aligned}u(x,t)&=\left\{\frac{4\varrho_1\vartheta_3^2\left(\delta_2-\delta_1\right)}{4\vartheta_3^2-\left[\left(\delta_1-\delta_2\right)\left(x+\left\{\frac{4\kappa\epsilon_1\mu_4+9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right)\right]^2}\right\}^{1/3}\\&\times\exp\left[i\left\{-\kappa x+\left(\frac{2\mu_4\left(10\kappa^3\left(b_1\kappa-a_1\right)+3\varrho_0\left(4\epsilon_6\lambda_1+5\epsilon_2\varrho_0\right)\right)-5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t+\theta\right\}\right],\end{aligned} \tag{108}$$

$$\begin{aligned}v(x,t)&=\lambda\left\{\frac{4\varrho_1\vartheta_3^2\left(\delta_2-\delta_1\right)}{4\vartheta_3^2-\left[\left(\delta_1-\delta_2\right)\left(x+\left\{\frac{4\kappa\epsilon_1\mu_4+9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right)\right]^2}\right\}^{1/3}\\&\times\exp\left[i\left\{-\kappa x+\left(\frac{2\mu_4\left(10\kappa^3\left(b_1\kappa-a_1\right)+3\varrho_0\left(4\epsilon_6\lambda_1+5\epsilon_2\varrho_0\right)\right)-5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t+\theta\right\}\right].\end{aligned} \tag{109}$$

However, when $\Theta(\psi)=(\psi-\delta_1)^2(\psi-\delta_2)^2$, $\varrho_0=-\varrho_1\delta_1$ and $s_0=0$, cubic–quartic singular solitons are:

$$\begin{aligned}u(x,t)&=\left\{\frac{\varrho_1\left(\delta_2-\delta_1\right)}{2}\left(1\mp\coth\left[\frac{\delta_1-\delta_2}{2\vartheta_3}\left(x+\left\{\frac{4\kappa\epsilon_1\mu_4+9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right)\right]\right)\right\}^{1/3}\\&\times\exp\left[i\left\{-\kappa x+\left(\frac{2\mu_4\left(10\kappa^3\left(b_1\kappa-a_1\right)+3\varrho_0\left(4\epsilon_6\lambda_1+5\epsilon_2\varrho_0\right)\right)-5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t+\theta\right\}\right],\end{aligned} \tag{110}$$

$$\begin{aligned}v(x,t)&=\lambda\left\{\frac{\varrho_1\left(\delta_2-\delta_1\right)}{2}\left(1\mp\coth\left[\frac{\delta_1-\delta_2}{2\vartheta_3}\left(x+\left\{\frac{4\kappa\epsilon_1\mu_4+9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right)\right]\right)\right\}^{1/3}\\&\times\exp\left[i\left\{-\kappa x+\left(\frac{2\mu_4\left(10\kappa^3\left(b_1\kappa-a_1\right)+3\varrho_0\left(4\epsilon_6\lambda_1+5\epsilon_2\varrho_0\right)\right)-5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t+\theta\right\}\right].\end{aligned} \tag{111}$$

Whenever $\Theta(\psi) = (\psi - \delta_1)^2(\psi - \delta_2)(\psi - \delta_3)$, $\delta_1 > \delta_2 > \delta_3$, $\varrho_0 = -\varrho_1\delta_1$ and $s_0 = 0$, cubic–quartic bright soliton is:

$$u(x,t) = \left\{\frac{\mathcal{D}_5}{\mathcal{F}_5 + \cosh\left[\mathcal{H}_7\left(x + \left\{\frac{4\kappa\epsilon_1\mu_4 + 9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right)\right]}\right\}^{1/3} \times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(10\kappa^3(b_1\kappa - a_1) + 3\varrho_0(4\epsilon_6\lambda_1 + 5\epsilon_2\varrho_0)\right) - 5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t + \theta\right\}\right], \tag{112}$$

$$v(x,t) = \lambda\left\{\frac{\mathcal{D}_5}{\mathcal{F}_5 + \cosh\left[\mathcal{H}_7\left(x + \left\{\frac{4\kappa\epsilon_1\mu_4 + 9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right)\right]}\right\}^{1/3} \times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(10\kappa^3(b_1\kappa - a_1) + 3\varrho_0(4\epsilon_6\lambda_1 + 5\epsilon_2\varrho_0)\right) - 5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t + \theta\right\}\right], \tag{113}$$

where

$$\mathcal{D}_5 = \frac{2\varrho_1(\delta_1 - \delta_2)(\delta_1 - \delta_3)}{\delta_3 - \delta_2}, \tag{114}$$

$$\mathcal{F}_5 = \frac{2\delta_1 - \delta_2 - \delta_3}{\delta_3 - \delta_2}, \tag{115}$$

$$\mathcal{H}_7 = \frac{\sqrt{(\delta_1 - \delta_2)(\delta_1 - \delta_3)}}{\vartheta_3}. \tag{116}$$

Here, $\mathcal{D}_5$ and $\mathcal{H}_7$ stand for the amplitude of soliton and its inverse width respectively. The condition $\varrho_1 < 0$ is necessary in order for the solitons to exist.

Finally, if $\Theta(\psi) = (\psi - \delta_1)(\psi - \delta_2)(\psi - \delta_3)(\psi - \delta_4)$, $\delta_1 > \delta_2 > \delta_3 > \delta_4$, $\varrho_0 = -\varrho_1\delta_2$ and $s_0 = 0$, JEF solutions are:

$$u(x,t) = \left\{\frac{\mathcal{D}_6}{\mathcal{F}_6 + \mathrm{sn}^2\left[\mathcal{H}_j\left(x + \left\{\frac{4\kappa\epsilon_1\mu_4 + 9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right), k\right]}\right\}^{1/3} \times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(10\kappa^3(b_1\kappa - a_1) + 3\varrho_0(4\epsilon_6\lambda_1 + 5\epsilon_2\varrho_0)\right) - 5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t + \theta\right\}\right], \tag{117}$$

$$v(x,t) = \lambda\left\{\frac{\mathcal{D}_6}{\mathcal{F}_6 + \mathrm{sn}^2\left[\mathcal{H}_j\left(x + \left\{\frac{4\kappa\epsilon_1\mu_4 + 9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right), k\right]}\right\}^{1/3} \times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(10\kappa^3(b_1\kappa - a_1) + 3\varrho_0(4\epsilon_6\lambda_1 + 5\epsilon_2\varrho_0)\right) - 5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t + \theta\right\}\right], \tag{118}$$

where

$$k = \frac{(\delta_2 - \delta_3)(\delta_1 - \delta_4)}{(\delta_1 - \delta_3)(\delta_2 - \delta_4)}, \tag{119}$$

$$\mathcal{D}_6 = \frac{\varrho_1(\delta_1 - \delta_2)(\delta_4 - \delta_2)}{\delta_1 - \delta_4}, \tag{120}$$

$$\mathcal{F}_6 = \frac{\delta_4 - \delta_2}{\delta_1 - \delta_4}, \tag{121}$$

$$\mathcal{H}_j = \frac{(-1)^j\sqrt{(\delta_1 - \delta_3)(\delta_2 - \delta_4)}}{2\vartheta_3} \text{ for } j = 8, 9. \tag{122}$$

Remark 5. When $k \to 1$, from (117) and (118), cubic–quartic singular optical solitons are obtained as

$$u(x,t) = \left\{ \frac{\mathcal{D}_6}{\mathcal{F}_6 + \tanh^2\left[\mathcal{H}_j\left(x + \left\{\frac{4\kappa\epsilon_1\mu_4 + 9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right)\right]} \right\}^{1/3} \times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(10\kappa^3(b_1\kappa - a_1) + 3\varrho_0(4\epsilon_6\lambda_1 + 5\epsilon_2\varrho_0)\right) - 5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t + \theta\right\}\right], \quad (123)$$

$$v(x,t) = \lambda\left\{ \frac{\mathcal{D}_6}{\mathcal{F}_6 + \tanh^2\left[\mathcal{H}_j\left(x + \left\{\frac{4\kappa\epsilon_1\mu_4 + 9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right)\right]} \right\}^{1/3} \times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(10\kappa^3(b_1\kappa - a_1) + 3\varrho_0(4\epsilon_6\lambda_1 + 5\epsilon_2\varrho_0)\right) - 5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t + \theta\right\}\right], \quad (124)$$

where $\delta_3 = \delta_4$.

Remark 6. For $k \to 0$, in this case, periodic singular solutions are

$$u(x,t) = \left\{ \frac{\mathcal{D}_6}{\mathcal{F}_6 + \sin^2\left[\mathcal{H}_j\left(x + \left\{\frac{4\kappa\epsilon_1\mu_4 + 9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right)\right]} \right\}^{1/3} \times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(10\kappa^3(b_1\kappa - a_1) + 3\varrho_0(4\epsilon_6\lambda_1 + 5\epsilon_2\varrho_0)\right) - 5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t + \theta\right\}\right], \quad (125)$$

$$v(x,t) = \lambda\left\{ \frac{\mathcal{D}_6}{\mathcal{F}_6 + \sin^2\left[\mathcal{H}_j\left(x + \left\{\frac{4\kappa\epsilon_1\mu_4 + 9\epsilon_2\epsilon_3\varrho_1^2\chi_0}{4b_1\mu_4}\right\}t\right)\right]} \right\}^{1/3} \times \exp\left[i\left\{-\kappa x + \left(\frac{2\mu_4\left(10\kappa^3(b_1\kappa - a_1) + 3\varrho_0(4\epsilon_6\lambda_1 + 5\epsilon_2\varrho_0)\right) - 5\epsilon_2\mu_2\varrho_1^2}{20\mu_4}\right)t + \theta\right\}\right], \quad (126)$$

where $\delta_2 = \delta_3$.

3. CONCLUSIONS

Today's paper successfully retrieved bright and singular CQ optical solitons to a brand new model. It is with Kudryashov's law of refractive index and that too with polarization mode dispersion. The rich and famous extended trial function approach made these solitons retrieval possible. The results were recovered for three integer values of the power law parameter n. One limitation of this approach is noticeably clear. The algorithm fails to recover dark soliton solutions. Nevertheless, the spectrum of soliton solutions thus recovered has yielded an abundance of opportunity to proceed further along in a variety of other avenues. An immediate thought is to recover the conservation laws for the model. One additional extension is to locate the governing model with DWDM topology and retrieve its soliton solutions along with the conservation laws. Yet another avenue is to study the model with fractional temporal evolution that has been successfully applied to complex Ginzburg–Landau equation [8]. Such studies are all under way and the results will be disseminated with time.

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CONFLICT OF INTEREST

The authors also declare that they have no conflicts of interest.

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