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ORIGINAL PAPER

Highly dispersive optical solitons in the nonlinear Schrödinger's equation having polynomial law of the refractive index change

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Abstract: In this paper, we apply the unified Riccati equation expansion method, as well as two forms of auxiliary equation methodology, to find highly dispersive optical solitons in the nonlinear Schrödinger's equation having a polynomial law of the refractive index change. Bright, dark and singular solitons as well as periodic and Jacobi elliptic solutions are obtained that are presented together with their existence criteria.

Keywords: Highly dispersive solitons; Polynomial law

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1. Introduction

One of the most important aspects of soliton science is the identification of exact soliton solutions. It is always a daunting task to locate an exact solution, especially when the governing model is highly nontrivial. This paper takes up the challenging task to attain that goal for the nonlinear Schrödinger equation with a polynomial law of the refractive index change. There exist several integration algorithms nowadays that make this achievement possible for a variety of physical situations [1–42]. This paper addresses highly dispersive optical solitons when the refractive index change stems from the polynomial law that is alternatively known as the cubic–quintic–septic (CQS) nonlinearity. A few effective integration methods will be implemented to the governing model that will reveal several forms of the optical solitons supported by the model.

Thus far, various ingenious integration algorithms have been invented for finding exact solutions to different nonlinear evolution equations. A few such exemplary schemes include the (G'/G)-expansion [18, 26, 29, 30], the modified simple equation method [31–33], generalized Kudryashov's algorithms [19, 34], various soliton ansatz methods [11, 35–38, 41], the new auxiliary equation methodology [13, 14], the exp $(-\phi(\xi))$-expansion [1, 12, 15, 24], the unified Riccati equation expansion [25], new auxiliary equation algorithms [23], the generalized auxiliary equation procedure [42], new extended auxiliary equation processes [39, 40], the semi-inverse variational principle [16, 17], the simplified Hirota's method [27, 28] and a variety of others.

A selected few algorithms will be implemented to address highly dispersive optical solitons in the present model. These concepts have been proposed in a number of

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recent articles [2–10]. More precisely, the governing nonlinear Schrödinger's equation (NLSE) with inter-modal dispersion (IMD), third-order dispersion (3OD), fourth-order dispersion (4OD), fifth-order dispersion (5OD) and sixth-order dispersion (6OD) is taken into consideration, in addition to the usual group velocity dispersion (GVD). The details of the mathematical model and its technicalities are exposed in the subsequent sections and subsections.

1.1. The governing model

The objective of this paper is to apply three integration algorithms mentioned above, to construct the optical solitons and other solutions of the following NLSE with CQS nonlinearity [8–10]:

$$\begin{aligned} &iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} \\ &\quad + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} \\ &\quad + \left(b_1|q|^2 + b_2|q|^4 + b_3|q|^6\right)q = 0, \end{aligned} \tag{1}$$

where the complex-valued function $q(x, t)$ is the wave function that represents solitons and other forms of nonlinear waves, while x and t are independent spatial and temporal variables, and $i = \sqrt{-1}$. The first term describes temporal evolution, while $a_j (j = 1, 2, \ldots, 6)$ are real constants which represent IMD, GVD, 3OD, 4OD, 5OD and 6OD, respectively. Next, b_1, b_2 and b_3 are the coefficients of the polynomial CQS nonlinearity, coming from the change in the refractive index of the medium. Equation (1) has been discussed in [8–10] by using the F-expansion method, $\exp(-\psi(\varsigma))$-expansion method and the extended Jacobi's elliptic function expansion.

The article is organized as follows: In Sect. 2, the mathematical analysis of Eq. (1) is performed. In Sect. 3, the preliminaries are discussed. In Sects. 4–6, Eq. (1) is solved using the three integration algorithms mentioned above. In Sect. 7, conclusions are given.

2. Mathematical analysis

For traveling wave solutions of Eq. (1), the form of the solution is presumed to be

$$q(x, t) = g(\xi) e^{i\psi(x,t)}, \tag{2}$$

where $g(\xi)$ is a real function representing the amplitude of the wave, with the traveling wave variable

$$\xi = x - vt \tag{3}$$

such that v is the group speed of the wave. The phase of the wave $\psi(x, t)$ is taken as

$$\psi(x, t) = -kx + \omega t + \theta, \tag{4}$$

where k is the frequency, ω is the wave number, and θ is the phase shift of the soliton. Substituting (2) into (1) and separating the real and imaginary parts, one finds that the real part has the form

$$\begin{aligned} &a_6 g^{(6)} + (a_4 - 5a_5 k - 15a_6 k^2) g^{(4)} + (a_2 + 3a_3 k \\ &\quad - 6a_4 k^2 - 10a_5 k^3 + 15a_6 k^4) g'' \\ &\quad + (-\omega + a_1 k - a_2 k^2 - a_3 k^3 + a_4 k^4 + a_5 k^5 - a_6 k^6) g \\ &\quad + b_1 g^3 + b_2 g^5 + b_3 g^7 = 0, \end{aligned} \tag{5}$$

while the imaginary part is

$$\begin{aligned} &(v - a_1 + 2a_2 k + 3a_3 k^2 - 4a_4 k^3 - 5a_5 k^4 \\ &\quad + 6a_6 k^5) g + (-a_3 + 4a_4 k + 10a_5 k^2 \\ &\quad - 20a_6 k^3) g'' - (a_5 - 6a_6 k) g^{(4)} = 0. \end{aligned} \tag{6}$$

The superscripts of g represent the derivatives w.r.t. ξ. From Eq. (6) and with aid of the principle of linear independence of different powers of k terms, one gets:

$$a_5 - 6ka_6 = 0 \tag{7}$$

$$3a_3 - 4k(3a_4 + 5a_5 k) = 0 \tag{8}$$

and the soliton speed is given by

$$v = a_1 - 2a_2 k - 8k^3(a_4 + 2a_5 k). \tag{9}$$

Now, real part (5) reduces to

$$\begin{aligned} &g^{(6)} + A_4 g^{(4)} + A_2 g'' + A_1 g \\ &\quad + A_3 g^3 + A_5 g^5 + A_7 g^7 = 0, \end{aligned} \tag{10}$$

where

$$\begin{aligned} A_1 &= -\frac{k}{a_5}\left(6\omega - 6a_1 k + 6a_2 k^2 + 18a_4 k^4 + 35a_5 k^5\right), \\ A_2 &= \frac{3k}{a_5}\left(2a_2 + 12a_4 k^2 + 25a_5 k^3\right), \\ A_3 &= \frac{6b_1 k}{a_5}, A_4 = \frac{3k}{a_5}(2a_4 + 5a_5 k), \\ A_5 &= \frac{6b_2 k}{a_5}, \\ A_7 &= \frac{6b_3 k}{a_5}, \end{aligned} \tag{11}$$

provided $a_5 \neq 0$.

The problem is reduced to solving Eq. (10). To this end, we use the proposed three methods mentioned in abstract.

3. Preliminaries

We further assume that Eq. (10) has the formal solution:

$$g(\xi)=\sum_{i=0}^{M}\alpha_i F^i(\xi), \tag{12}$$

where $F(\xi)$ satisfies the first-order auxiliary equation:

$$[F'(\xi)]^\tau=\sum_{j=0}^{N}h_j F^j(\xi). \tag{13}$$

Here α_i and h_j are constants to be determined such that $\alpha_M\neq0$ and $h_N\neq0$; M, N are related positive integers, and $\tau=1$ or 2. The choice of the solution method crucially depends on τ. We determine M in (12) by using the homogeneous balance method, as follows:

If $D(g)=M$, then $D(g')=M+\frac{N}{\tau}-1, D(g'')=M+\frac{2N}{\tau}-2$, and hence we have the relation

$$D\left[g^s g^{(r)}\right]=M(s+1)+\left(\frac{N}{\tau}-1\right)r. \tag{14}$$

Balancing $g^{(6)}$ with g^7 in Eq. (10) and using (14), one gets:

$$M=\frac{N}{\tau}-1. \tag{15}$$

In the next sections, we will solve Eq. (1) utilizing different methods, depending on the value of τ.

4. Unified Riccati expansion

If $\tau=1$ in Eq. (15), then one gets $M=N-1$. By choosing $N=2$, one gets $M=1$. Then, according to the unified Riccati equation expansion method [25], Eq. (10) has the formal solution:

$$g(\xi)=\alpha_0+\alpha_1F(\xi), \tag{16}$$

where α_0 and α_1 are constants to be determined, such that $\alpha_1\neq0$ and $F(\xi)$ satisfies the Riccati equation:

$$F'(\xi)=h_0+h_1F(\xi)+h_2F^2(\xi). \tag{17}$$

Here h_0, h_1 and h_2 are constants to be determined such that $h_2\neq0$. Substituting (16) along with (17) into Eq. (10), collecting all the coefficients of $F^j(\xi)$ $(j=0,1,\ldots,7)$, one finds the following set of algebraic equations:

$$\begin{aligned}
&F^7(\xi): A_7\alpha_1^7+720\alpha_1h_2^6=0,\\
&F^6(\xi): 2520\alpha_1h_2^5h_1+7A_7\alpha_0\alpha_1^6=0,\\
&F^5(\xi): 24A_4\alpha_1h_2^4+A_5\alpha_1^5\\
&\quad+1680\alpha_1h_2^5h_0+3360\alpha_1h_2^4h_1^2+21A_7\alpha_0^2\alpha_1^5=0,\\
&F^4(\xi): 60A_4\alpha_1h_2^3h_1+4200\alpha_1h_2^4h_1h_0+35A_7\alpha_0^3\alpha_1^4\\
&\quad+2100\alpha_1h_2^3h_1^3+5A_5\alpha_0\alpha_1^4=0,\\
&F^3(\xi): 10A_5\alpha_0^3\alpha_1^2+1848\alpha_1h_2^3h_1h_0^2+60A_4\alpha_1h_2^2h_0h_1\\
&\quad+63\alpha_1h_2h_1^5+3A_2\alpha_1h_1h_2\\
&\quad+15A_4\alpha_1h_1^3h_2+1176\alpha_1h_2^2h_1^3h_0+21A_7\alpha_0^5\alpha_1^2\\
&\quad+3A_3\alpha_0\alpha_1^2=0,\\
&F^2(\xi): 50A_4\alpha_1h_1^2h_2^2+602\alpha_1h_2^2h_1^4\\
&\quad+35A_7\alpha_0^4\alpha_1^3+3584\alpha_1h_2^3h_1^2h_0+10A_5\alpha_0^2\alpha_1^3\\
&\quad+2A_2\alpha_1h_2^2+1232\alpha_1h_2^4h_0^2\\
&\quad+40A_4\alpha_1h_2^3h_0+A_3\alpha_1^3=0,\\
&F^1(\xi): A_1\alpha_0+A_3\alpha_0^3+A_5\alpha_0^5+A_7\alpha_0^7+A_4\alpha_1h_1^3h_0\\
&\quad+A_2\alpha_1h_1h_0+52\alpha_1h_2h_1^3h_0^2\\
&\quad+136\alpha_1h_2^2h_0^3h_1+8A_4\alpha_1h_2h_0^2h_1\\
&\quad+\alpha_1h_1^5h_0=0,\\
&F^0(\xi): A_4\alpha_1h_1^4+A_2\alpha_1h_1^2+2A_2\alpha_1h_0h_2+7A_7\alpha_0^6\alpha_1\\
&\quad+5A_5\alpha_0^4\alpha_1+22A_4\alpha_1h_2h_0h_1^2\\
&\quad+3A_3\alpha_0^2\alpha_1\\
&\quad+16A_4\alpha_1h_2^2h_0^2+\alpha_1h_1^6+A_1\alpha_1+720\alpha_1h_2^2h_1^2h_0^2\\
&\quad+114\alpha_1h_2h_1^4h_0+272\alpha_1h_2^3h_0^3=0.
\end{aligned} \tag{18}$$

On solving the above algebraic Eq. (18) by using the Maple, we find the following results:

$$\begin{aligned}
A_1=&\frac{A_4}{8575A_7}\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\\
&\left[2A_4A_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}+\frac{75A_5^2}{A_7}-245A_3\right]\\
&-\frac{1}{8575A_7^2}\left(900A_5^3-3675A_3A_5A_7-18A_4^3A_7^2\right),\\
A_2=&\frac{1}{175}\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\\
&\left[A_4A_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}-175A_3+55\frac{A_5^2}{A_7}\right]+\frac{4}{25}A_4^2,\\
&\alpha_0=0, \alpha_1=h_2\left(-\frac{720}{A_7}\right)^{\frac{1}{6}},\\
&h_0=-\frac{1}{420h_2}\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}+6A_4\right], h_1=0, h_2=h_2,
\end{aligned} \tag{19}$$

provided

$$A_7<0. \tag{20}$$

It is well known [25] that Riccati equation (17) has the following fractional solutions:

$$F(\xi)=\begin{cases}-\dfrac{h_1}{2h_2}-\dfrac{\sqrt{\Delta}\left[r_1\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)+r_2\right]}{2h_2\left[r_2\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)+r_1\right]}, & \text{if } \Delta>0 \text{ and } r_1^2+r_2^2\neq 0,\\ -\dfrac{h_1}{2h_2}+\dfrac{\sqrt{-\Delta}\left[r_3\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)-r_4\right]}{2h_2\left[r_4\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)+r_3\right]}, & \text{if } \Delta<0 \text{ and } r_3^2+r_4^2\neq 0,\\ -\dfrac{h_1}{2h_2}+\dfrac{1}{h_2\xi+r_5}, & \text{if } \Delta=0,\end{cases} \tag{21}$$

where $r_i (i=1,2,\ldots,5)$ are arbitrary constants. Here $\Delta = h_1^2-4h_0h_2$ can be written in the form:

$$\Delta=\frac{1}{105}\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}+6A_4\right]. \tag{22}$$

By the aid of solution (21), we find the following types of solutions for Eq. (1):

Type 1: $\Delta>0$. Substituting (19) along with (21) into Eq. (16), we have the solitary wave solutions of Eq. (1) in the form:

$$q(x,t)=\pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{\Delta}\left[\frac{r_1\tanh\left(\frac{1}{2}[x-a_1+2a_2k+8k^3(a_4+2a_5k)]\sqrt{\Delta}\right)+r_2}{r_2\tanh\left(\frac{1}{2}[x-a_1+2a_2k+8k^3(a_4+2a_5k)]\sqrt{\Delta}\right)+r_1}\right]\exp[i(-kx+\omega t+\theta].\tag{23}$$

In particular if $r_1\neq 0$ and $r_2=0$, then Eq. (1) has the dark soliton solutions of the form:

$$q(x,t)=\pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{\Delta}\left[\tanh\left([x-a_1+2a_2k+8k^3(a_4+2a_5k)]\frac{1}{2}\sqrt{\Delta}\right)\right]\exp[i(-kx+\omega t+\theta],\tag{24}$$

while, if $r_1=0$ and $r_2\neq 0$, it has the singular soliton solutions of the form:

$$q(x,t)=\pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{\Delta}\left[\coth\left([x-a_1+2a_2k+8k^3(a_4+2a_5k)]\frac{1}{2}\sqrt{\Delta}\right)\right]\exp[i(-kx+\omega t+\theta].\tag{25}$$

Type 2: $\Delta<0$. Substituting (19) along with (21) into Eq. (16), we have the periodic wave solutions of Eq. (1) in the form:

$$q(x,t)=\pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{-\Delta}\left[\frac{r_3\tan\left(\frac{1}{2}[x-a_1+2a_2k+8k^3(a_4+2a_5k)]\sqrt{-\Delta}\right)-r_4}{r_4\tan\left(\frac{1}{2}[x-a_1+2a_2k+8k^3(a_4+2a_5k)]\sqrt{-\Delta}\right)+r_3}\right]\exp[i(-kx+\omega t+\theta],\tag{26}$$

In particular if $r_3\neq 0$ and $r_4=0$, then Eq. (1) has the periodic wave solutions of the form:

$$q(x,t)=\pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{-\Delta}\left[\tan\left([x-a_1+2a_2k+8k^3(a_4+2a_5k)]\frac{1}{2}\sqrt{-\Delta}\right)\right]\exp[i(-kx+\omega t+\theta],\tag{27}$$

while, if $r_3=0$ and $r_4\neq 0$, it has the periodic wave solutions of the form:

$$q(x,t)=\pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{-\Delta}\left[\cot\left([x-a_1+2a_2k+8k^3(a_4+2a_5k)]\frac{1}{2}\sqrt{-\Delta}\right)\right]\exp[i(-kx+\omega t+\theta],\tag{28}$$

On using (11) and (19) one can show that the wave number ω in (23)–(28) is given by:

$$\omega = \frac{1}{51450k}\left\{8575a_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\left[\left(\frac{1}{35}A_4 + k^2\right)A_3 - \frac{11A_5^2}{35A_7}\left(k^2 + \frac{15}{539}A_4\right)\right] + \frac{75a_5A_5}{A_7^2}\left(12A_5^2 - 49A_3A_7\right) - \left[2a_5\left(9A_4^3 - 171500k^6 + 686k^2A_4^2\right) - 154350k^2\left(k^3a_4 + \frac{1}{3}a_1\right)\right] - 49A_4a_5A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}\left(k^2 + \frac{2}{49}A_4\right)\right\}, \tag{29}$$

where

$$a_5 = \frac{1050k(a_2 + 6k^2a_4)}{\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\left[A_4A_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} - 175A_3 + \frac{55A_5^2}{A_7}\right] - 13125k^4 + 28A_4^2}, \tag{30}$$

in which $A_i (i = 3, 4, 5, 7)$ are given by (11).

Type 3: $\Delta = 0$. Then $h_1^2 = 4h_0h_2$ and consequently, algebraic Eq. (18) gives the results:

$$A_1 = 0, A_2 = -A_3\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}, A_5 = -\frac{6A_4}{\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}},$$
$$\alpha_0 = 0, \alpha_1 = h_2\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}, h_0 = 0, h_1 = 0, h_2 = h_2. \tag{31}$$

Substituting (31) along with (21) into Eq. (16), we have the rational solutions of Eq. (1) in the form:

$$q(x,t) = \pm\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\left[\frac{h_2}{h_2\xi + r_5}\right]\exp[i(-kx + \omega t + \theta], \tag{32}$$

On using (11) and (31) one can show that the wave number ω of the soliton in (32) is given by:

$$\omega = \frac{k}{6}\left[40k^4a_5 + 6a_1 + A_3\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}a_5 + 18k^3a_4\right], \tag{33}$$

and

$$b_2 = -\frac{A_4a_5}{k}\left(-\frac{90}{A_7}\right)^{-\frac{2}{3}}, \tag{34}$$

where

$$a_5 = -\frac{6k(a_2 + 6k^2a_4)}{A_3\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} + 75k^4}, \tag{35}$$

in which $A_i (i = 3, 7)$ are given by (11).

5. The new auxiliary equation

If $\tau = 2$ in Eq. (15), then one has $M = \frac{N}{2} - 1$. According to this method [23], if we choose $N = 8$, then we find $M = 3$. In this case Eq. (10) has the formal solution:

$$g(\xi) = \alpha_0 + \alpha_1F(\xi) + \alpha_2F^2(\xi) + \alpha_3F^3(\xi), \tag{36}$$

where $\alpha_0, \alpha_1, \alpha_2$ and α_3 are constants to be determined, such that $\alpha_3 \neq 0$ and $F(\xi)$ satisfies the new auxiliary equation:

$$F'^2(\xi) = \sum_{j=0}^{8} h_jF^j(\xi). \tag{37}$$

Here $h_j (j = 0, 1, \ldots, 8)$ are constants to be determined such that $h_8 \neq 0$. Substituting (36) along with (37) into Eq. (10), collecting all the coefficients of $F^j(\xi)[F'(\xi)]^k$ $(j = 0, 1, \ldots, 21, k = 0, 1)$, one obtains a set of algebraic equations which can be solved by using the Maple to get the following sets of solutions:

Set 1.

$$A_1 = \frac{A_4}{8575}\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\left[12A_4A_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} - \frac{45A_5^2}{2A_7} - 245A_3\right] - \frac{1}{1715A_7^2}\left(135A_5^3 - 735A_3A_5A_7 - 9A_4^3A_7^2\right),$$
$$A_2 = \frac{1}{350}\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\left[8A_4A_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} - 350A_3 + \frac{65A_5^2}{A_7}\right] + \frac{74}{350}A_4^2,$$
$$\alpha_3 = 3\left(-\frac{720h_8^3}{A_7}\right)^{\frac{1}{6}}, \alpha_2 = 0, \alpha_1 = 0,$$
$$\alpha_0 = 0, h_0 = h_1 = h_3 = h_4 = h_5 = h_6 = h_7 = 0,$$
$$h_2 = -\frac{1}{1890}\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}} + 6A_4\right], \tag{38}$$

provided

$$A_7 < 0 \text{ and } h_8 > 0. \tag{39}$$

Set 2.

$$A_1 = \frac{A_4}{8575}\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} \left[2A_4A_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} + \frac{75A_5^2}{A_7} - 245A_3\right] + \frac{1}{8575A_7^2} \left[3675A_3A_5A_7 - 900A_5^3 + 18A_4^3A_7^2\right],$$
$$A_2 = \frac{1}{175}\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} \left[A_4A_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} - 175A_3 + \frac{55A_5^2}{A_7}\right] + \frac{28}{175}A_4^2, \tag{40}$$
$$\alpha_3 = 3\left(-\frac{720h_8^3}{A_7}\right)^{\frac{1}{6}}, \quad \alpha_2 = \alpha_1 = 0,$$
$$\alpha_0 = \left(-\frac{90}{A_7}\right)^{\frac{1}{6}} \sqrt{\frac{1}{210}\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}} + 6A_4\right]},$$
$$h_2 = \frac{1}{945}\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}} + 6A_4\right],$$
$$h_5 = \frac{2}{315}\sqrt{105h_8\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}} + 6A_4\right]},$$
$$h_0 = h_1 = h_3 = h_4 = h_6 = h_7 = 0,$$

provided

$$A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}} + 6A_4 > 0, A_7 < 0 \text{ and } h_8 > 0. \tag{41}$$

According to the well-known solutions of Eq. (37) obtained in [23], we find the following types of solutions:

Type 1: If $h_0 = h_1 = h_3 = h_4 = h_6 = h_7 = 0$, $h_5^2 - 4h_2h_8 > 0$, $h_2 > 0$, substituting (38) into Eq. (36), we have the periodic wave solutions of Eq. (1) in the form:

$$q(x,t) = \pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{2\Delta} \left[\sec\left(\frac{1}{2}\left[x - a_1 + 2a_2k + 8k^3(a_4 + 2a_5k)\right]\sqrt{2\Delta}\right)\right] \exp[i(-kx + \omega t + \theta], \tag{42}$$

provided

$$\Delta = \frac{1}{105}\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}} + 6A_4\right] > 0. \tag{43}$$

Type 2: If $h_0 = h_1 = h_3 = h_4 = h_6 = h_7 = 0$, $h_5^2 - 4h_2h_8 < 0$, $h_2 > 0$, substituting (38) into Eq. (36), we find the singular soliton solutions of Eq. (1) in the form:

$$q(x,t) = \pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{-2\Delta} \left[\operatorname{csch}\left(\frac{1}{2}\left[x - a_1 + 2a_2k + 8k^3(a_4 + 2a_5k)\right]\sqrt{-2\Delta}\right)\right] \exp[i(-kx + \omega t + \theta], \tag{44}$$

provided

$$\Delta = \frac{1}{105}\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}} + 6A_4\right] < 0. \tag{45}$$

On using (11) and (38), one can show that the wave number ω in (42) and (44) is given by:

$$\omega = \frac{1}{51450k}\left\{5a_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\left[49A_3(A_4 + 35k^2) - \frac{A_5^2}{2A_7}(637k^2 - 9A_4)\right] - \frac{75a_5A_5}{A_7^2}(49A_3A_7 - 9A_5^2) + a_5(343000k^6 - 45A_4^3 - 1813k^2A_4^2) + 154350k^2\left(k^3a_4 + \frac{1}{3}a_1\right) - 4a_5A_4A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}(49k^2 + 3A_4)\right\}, \tag{46}$$

where

$$a_5 = \frac{1050k(a_2 + 6k^2a_4)}{\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\left[4A_4A_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} - 175A_3 + \frac{65A_5^2}{2A_7}\right] - 13125k^4 + 37A_4^2}, \tag{47}$$

in which $A_i (i = 3, 4, 5, 7)$ are given by (11).

Type 3. If $h_0 = h_1 = h_3 = h_4 = h_6 = h_7 = 0, h_5^2 - 4h_2h_8 = 0, h_2 > 0$, substituting (40) into Eq. (36), we have the same dark soliton solution (24) and the same singular soliton solution (25).

6. The new extended auxiliary equation

According to the new extended auxiliary equation method [39, 40], Eq. (10) may have the formal solution:

$$g(\xi) = \alpha_0 + \alpha_1 F(\xi) + \alpha_2 F^2(\xi), \tag{48}$$

where α_0, α_1 and α_2 are constants to be determined such that $\alpha_2 \neq 0$ and the function $F(\xi)$ satisfies the following first-order auxiliary equation

$$F'^2(\xi) = h_0 + h_2F^2(\xi) + h_4F^4(\xi) + h_6F^6(\xi), \tag{49}$$

such that $h_j (j = 0, 2, 4, 6)$ are constants to be determined. It is well known that Eq. (49) has the following solution:

$$F(\xi)=\frac{1}{2}\left[-\frac{h_4}{h_6}(1\pm f(\xi))\right]^{\frac{1}{2}}, \tag{50}$$

where $f(\xi)$ could be expressed through the Jacobi elliptic functions sn (ξ,m), cn (ξ,m), dn (ξ,m) and so on. Here $0<m<1$ is the modulus of the Jacobi elliptic functions. Substituting (48) along with (49) into Eq. (10), collecting the coefficients of each power $F^i(\xi)(F'(\xi))^j, (i=0,1,2,\ldots,14, j=0,1)$, and setting these coefficients to zero, one obtains the following system of algebraic equations:

$$\begin{aligned}
&[F(\xi)]^{14}\colon 46080\alpha_2h_6^3+A_7\alpha_2^7=0,\\
&[F(\xi)]^{13}\colon 7A_7\alpha_1\alpha_2^6+10395\alpha_1h_6^3=0,\\
&[F(\xi)]^{12}\colon 7A_7\alpha_0\alpha_2^6+80640\alpha_2h_6^2h_4+21A_7\alpha_1^2\alpha_2^5=0,\\
&[F(\xi)]^{11}\colon 17010\alpha_1h_6^2h_4+35A_7\alpha_1^3\alpha_2^4+42A_7\alpha_0\alpha_1\alpha_2^5=0,\\
&[F(\xi)]^{10}\colon (A_5+21A_7\alpha_0^2)\alpha_2^5+105A_7\alpha_0\alpha_1^2\alpha_2^4\\
&\quad+35A_7\alpha_1^4\alpha_2^3\\
&\quad+40320h_6\left[\left(\frac{1}{105}A_4+\frac{4}{3}h_2\right)h_6+h_4^2\right]\alpha_2=0,\\
&[F(\xi)]^{9}\colon 7560\alpha_1\left\{\left(\frac{1}{72}A_7\alpha_0^2+\frac{1}{1512}A_5\right)\alpha_2^4\right.\\
&\quad+\frac{1}{54}\alpha_2^3A_7\alpha_0\alpha_1^2+\frac{1}{360}\alpha_2^2A_7\alpha_1^4\\
&\quad\left.+\left[\left(\frac{1}{72}A_4+\frac{107}{72}h_2\right)h_6+h_4^2\right]h_6\right\}=0,\\
&[F(\xi)]^{8}\colon 5040\alpha_2\left\{\frac{1}{144}\left(A_7\alpha_0^2+\frac{1}{7}A_5\right)\alpha_0\alpha_2^3\right.\\
&\quad+\frac{1}{24}\alpha_1^2\left(\frac{1}{21}A_5+A_7\alpha_0^2\right)\alpha_2^2\\
&\quad+\frac{1}{48}\alpha_2A_7\alpha_0\alpha_1^4+\frac{1}{720}A_7\alpha_1^6+8h_0h_6^2\\
&\quad\left.+\frac{2}{21}h_4h_6(98h_2+A_4)+h_4^3\right\}=0,\\
&[F(\xi)]^{7}\colon 720\left\{\frac{7}{36}\left(A_7\alpha_0^2+\frac{1}{7}A_5\right)\alpha_0\alpha_2^3\right.\\
&\quad+\frac{7}{24}\alpha_1^2\left(\frac{1}{21}A_5+A_7\alpha_0^2\right)\alpha_2^2\\
&\quad+\frac{7}{120}\alpha_2A_7\alpha_0\alpha_1^4+\frac{1}{720}A_7\alpha_1^6+\frac{47}{4}h_0h_6^2\\
&\quad\left.+\frac{1}{6}h_4(A_4+71h_2)h_6+h_4^3\right\}\alpha_1=0,\\
&[F(\xi)]^{6}\colon (10A_5\alpha_0^2+35A_7\alpha_0^4+A_3)\alpha_2^3\\
&\quad+210\alpha_1^2\left(A_7\alpha_0^2+\frac{1}{7}A_5\right)\alpha_0\alpha_2^2+7A_7\alpha_0\alpha_1^6\\
&\quad+\left[(105A_7\alpha_0^2+5A_5)\alpha_1^4+(8A_2+32256h_0h_4\right.\\
&\quad+11648h_2^2\\
&\quad\left.+320h_2A_4)h_6+120h_4^2(56h_2+A_4)\right]\alpha_2=0,\\
&[F(\xi)]^{5}\colon 24\alpha_1\left\{\left(\frac{5}{4}A_5\alpha_0^2+\frac{35}{8}A_7\alpha_0^4+\frac{1}{8}A_3\right)\alpha_2^2\right.\\
&\quad+\frac{35}{6}\alpha_1^2\left(A_7\alpha_0^2+\frac{1}{7}A_5\right)\alpha_0\alpha_2+\left(\frac{1}{24}A_5+\frac{7}{8}A_7\alpha_0^2\right)\alpha_1^4\\
&\quad+\left(\frac{1}{8}A_2+\frac{483}{2}h_0h_4+\frac{651}{8}h_2^2+\frac{13}{4}h_2A_4\right)h_6\\
&\quad\left.+h_4^2(A_4+35h_2)\right\}=0,\\
&[F(\xi)]^{4}\colon \alpha_2\left[(30A_5\alpha_0^2+3A_3+105A_7\alpha_0^4)\alpha_1^2\right.\\
&\quad+240h_0(56h_2+A_4)h_6\\
&\quad\left.+4032\left(\frac{1}{2}h_2^2+h_0h_4+\frac{5}{168}h_2A_4+\frac{1}{672}A_2\right)h_4\right]\\
&\quad+35\alpha_1^4\left(A_7\alpha_0^2+\frac{1}{7}A_5\right)\alpha_0+(21A_7\alpha_0^5+3A2_3\alpha_0\\
&\quad+10A_5\alpha_0^3)\alpha_2^2=0,\\
&[F(\xi)]^{3}\colon 504\alpha_1\left\{\frac{1}{12}\alpha_0\left(A_7\alpha_0^4+\frac{1}{7}A_3+\frac{10}{21}A_5\alpha_0^2\right)\alpha_2\right.\\
&\quad+\left(\frac{5}{252}A_5\alpha_0^2+\frac{5}{72}A_7\alpha_0^4+\frac{1}{504}A_3\right)\alpha_1^2\\
&\quad+\frac{5}{42}h_0(A_4+35h_2)h_6\\
&\quad\left.+\left(\frac{5}{126}h_2A_4+\frac{1}{252}A_2+h_0h_4+\frac{13}{36}h_2^2\right)h_4\right\}=0,\\
&[F(\xi)]^{2}\colon \left[7A_7\alpha_0^6+5A_5\alpha_0^4+3A_3\alpha_0^2+2880h_0^2h_6\right.\\
&\quad+72h_0(24h_2+A_4)h_4\\
&\quad\left.+64h_2^3+4h_2A_2+A_1+16h_2^2A_4\right]\alpha_2\\
&\quad+21\alpha_1^2\alpha_0\left(A_7\alpha_0^4+\frac{1}{7}A_3+\frac{10}{21}A_5\alpha_0^2\right)=0,\\
&[F(\xi)]^{1}\colon 12\alpha_1\left[\frac{7}{12}A_7\alpha_0^6+\frac{5}{12}A_5\alpha_0^4+\frac{1}{4}A_3\alpha_0^2+30h_0^2h_6\right.\\
&\quad+h_0(11h_2+A_4)h_4+\frac{1}{12}h_2A_2+\frac{1}{12}A_1\\
&\quad\left.+\frac{1}{12}h_2^2A_4+\frac{1}{12}h_2^3\right]=0,\\
&[F(\xi)]^{0}\colon 144h_0\left(h_0h_4+\frac{2}{9}h_2^2+\frac{1}{72}A_2+\frac{1}{18}h_2A_4\right)\alpha_2\\
&\quad+\alpha_0(A_1+A_3\alpha_0^2+A_5\alpha_0^4+A_7\alpha_0^6)=0.
\end{aligned} \tag{51}$$

According to [39, 40], we have the following types of solutions:

Type 1: If $h_0=\dfrac{h_4^3(m^2-1)}{32h_6^2m^2}$, $h_2=\dfrac{h_4^2(5m^2-1)}{16h_6m^2}$, $h_6>0$, then from the above algebraic Eq. (51) and by using the Maple, one gets the results:

$$
\begin{aligned}
A_1 &= \frac{1}{17150(m^2+1)^2A_7^2}\{[24A_4A_5A_7 \\
&\left(m^4-\frac{4}{3}m^2+1\right)\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} \\
&\quad -45\left(m^4-\frac{46}{3}m^2+1\right)A_5^2 \\
&\quad -490(m^2+1)^2A_3A_7\Big]A_4A_7\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} \\
&\quad -1350(m^2+3)\left(m^2+\frac{1}{3}\right)A_5^3 \\
&\quad +7350(m^2+1)^2A_3A_5A_7 \\
&\quad +90\left(m^4-\frac{2}{5}m^2+1\right)A_4^3A_7^2\Big\}, \\
A_2 &= \frac{1}{350(m^2+1)^2A_7}\{[8A_4A_5A_7 \\
&\quad (m^4-m^2+1)\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} \\
&\quad -350(m^2+1)^2A_3A_7 \\
&\quad +65\left(m^4+\frac{62}{13}m^2+1\right)A_5^2\Big] \\
&\quad \times\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}+74\left(m^4+\frac{38}{37}m^2+1\right)A_4^2A_7\Big\}, \\
&\quad \alpha_2 = 2\left(-\frac{720h_6^3}{A_7}\right)^{\frac{1}{6}}, \quad \alpha_1 = 0, \\
\alpha_0 &= m\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{\frac{A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}+6A_4}{210(m^2+1)}}, \\
h_4 &= 2m\sqrt{\frac{h_6\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}+6A_4\right]}{210(m^2+1)}}, h_6 = h_6,
\end{aligned} \tag{52}
$$

provided

$$
A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}+6A_4 > 0, A_7<0 \text{ and } h_6 > 0. \tag{53}
$$

Substituting (52) into Eq. (48), we have the Jacobi elliptic function solutions of Eq. (1) in the form:

$$
\begin{aligned}
q(x,t) &= \pm\frac{m}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{\frac{2}{(m^2+1)}\Delta} \\
&\Bigg[\operatorname{sn}\left(\frac{1}{2}[x-a_1+2a_2k\right. \\
&\left.+8k^3(a_4+2a_5k)]\sqrt{\frac{2}{(m^2+1)}\Delta}\right)\Bigg] \\
&\exp[i(-kx+\omega t+\theta],
\end{aligned} \tag{54}
$$

or

$$
\begin{aligned}
q(x,t) &= \pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{\frac{2}{(m^2+1)}\Delta} \\
&\Bigg[\operatorname{ns}\left(\frac{1}{2}[x-a_1+2a_2k\right. \\
&\left.+8k^3(a_4+2a_5k)]\sqrt{\frac{2}{(m^2+1)}\Delta}\right)\Bigg] \\
&\exp[i(-kx+\omega t+\theta],
\end{aligned} \tag{55}
$$

provided same constraint condition (43) is satisfied. On using (11) and (52), one can show that the wave number ω in (54) and (55) is given by:

$$
\begin{aligned}
\omega &= \frac{1}{51450(m^2+1)^2A_7^2k}\left\{-4a_5A_4A_5A_7^2\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}\right. \\
&\quad \left[m^4(3A_4+49k^2)-A_4(4m^2-3)\right. \\
&\quad \left.-49k^2(m^2-1)\right]+8575A_7a_5 \\
&\quad \times\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\left\{(m^2+1)^2A_3A_7\left(\frac{1}{35}A_4+k^2\right)\right. \\
&\quad -\frac{13}{70}A_5^2\left[m^4\left(k^2-\frac{9}{637}A_4\right)+m^2\left(\frac{62}{13}k^2+\frac{138}{637}A_4\right)\right. \\
&\quad \left.\left.+k^2-\frac{9}{637}A_4\right]\right\} \\
&\quad -343000A_7^2\left\{\left[(m^4+1)\left(\frac{9}{68600}A_4^3-k^6+\frac{37}{7000}k^2A_4^2\right)\right.\right. \\
&\quad \left.+m^2\left(\frac{19}{3500}k^2A_4^2-\frac{9}{171500}A_4^3-2k^6\right)\right]a_5 \\
&\quad \left.-\frac{9}{20}(m^2+1)^2k^2\left(k^3a_4+\frac{1}{3}a_1\right)\right\} \\
&\quad -3675A_3A_5A_7a_5(m^2+1)^2 \\
&\quad \left.+675a_5A_5^3(m^2+3)\left(m^2+\frac{1}{3}\right)\right\},
\end{aligned} \tag{56}
$$

where
in which $A_i(i=3,4,5,7)$ are given by (11).

$$a_5 = \frac{2100(m^2+1)^2 k(a_2+6a_4k^2)}{\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\left[5(m^4+2m^2+1)\left(\frac{13A_5^2}{A_7}-70A_3\right)+\frac{180m^2A_5^2}{A_7}+8(m^4-m^2+1)A_4A_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\right]-26250k^4(m^2+1)^2+74A_4^2\left(m^4+\frac{38}{37}m^2+1\right)}, \tag{57}$$

In particular, if $m \to 1$, then $sn(\xi,1) \to \tanh(\xi)$ and $ns(\xi,1) \to \coth(\xi)$. In this case from (54) and (55), we have the dark soliton solutions of Eq. (1) in the form:

$$q(x,t) = \pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{\Delta} \left[\tanh\left(\left[x-a_1+2a_2k+8k^3(a_4+2a_5k)\right]\frac{1}{2}\sqrt{\Delta}\right)\right] \exp[i(-kx+\omega t+\theta], \tag{58}$$

and the singular soliton solutions in the form:

$$q(x,t) = \pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{\Delta} \left[\coth\left(\left[x-a_1+2a_2k+8k^3(a_4+2a_5k)\right]\frac{1}{2}\sqrt{\Delta}\right)\right] \exp[i(-kx+\omega t+\theta]. \tag{59}$$

If $m \to 0$, then $ns(\xi,0) \to \csc(\xi)$. In this case from (55), we have the periodic solutions of Eq. (1) in the form:

$$q(x,t) = \pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{2\Delta} \left[\csc\left(\frac{1}{2}\left[x-a_1+2a_2k+8k^3(a_4+2a_5k)\right]\sqrt{2\Delta}\right)\right] \exp[i(-kx+\omega t+\theta]. \tag{60}$$

Type 2: If $h_0 = \frac{h_4^3(1-m^2)}{32h_6^2}$, $h_2 = \frac{h_4^2(5-m^2)}{16h_6}$, $h_6 > 0$, then from the above algebraic Eq. (51) and by using the Maple, one gets the results:

$$\begin{aligned}
A_1 &= \frac{1}{17150(m^2+1)^2A_7^2}\{[24A_4A_5A_7 \left(m^4-\frac{4}{3}m^2+1\right)\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} \\
&\quad - 45\left(m^4-\frac{46}{3}m^2+1\right)A_5^2 \\
&\quad -490(m^2+1)^2A_3A_7\Big]A_4A_7\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} \\
&\quad - 1350(m^2+3)\left(m^2+\frac{1}{3}\right)A_5^3 \\
&\quad + 7350(m^2+1)^2A_3A_5A_7 \\
&\quad +90\left(m^4-\frac{2}{5}m^2+1\right)A_4^3A_7^2\Big\}, \\
A_2 &= \frac{1}{350(m^2+1)^2A_7}\{[8A_4A_5A_7 (m^4-m^2-1)\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} \\
&\quad - 350(m^2+1)^2A_3A_7 \\
&\quad +65\left(m^4+\frac{62}{13}m^2+1\right)A_5^2\Big] \\
&\quad \times\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}+74\left(m^4+\frac{38}{37}m^2+1\right)A_4^2A_7\Big\}, \\
\alpha_2 &= 2\left(-\frac{720h_6^3}{A_7}\right)^{\frac{1}{6}}, \alpha_1 = 0, \\
\alpha_0 &= \left(-\frac{720}{A_7}\right)^{\frac{1}{6}} \sqrt{\frac{A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}+6A_4}{210(m^2+1)}}, \\
h_4 &= 2\sqrt{\frac{\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}+6A_4\right]h_6}{210(m^2+1)}}, h_6 = h_6,
\end{aligned} \tag{61}$$

provided same constraint condition (53) is satisfied. Substituting (61) into Eq. (48), then Eq. (1) has the same Jacobi elliptic function solutions (54) and (55).

Type 3: If $h_0 = \frac{h_4^3}{32h_6^2(1-m^2)}$, $h_2 = \frac{h_4^2(4m^2-5)}{16h_6(m^2-1)}$, $h_6 > 0$, then from the above algebraic Eq. (51), one gets the results:

$$\begin{aligned}
A_1 &= \frac{1}{17150(2m^2-1)^2A_7^2}\{[16A_4A_5A_7 \left(m^4-m^2+\frac{3}{2}\right)\left(-\frac{90}{A_7}\right)^{\frac{1}{3}} \\
&\quad + 600\left(m^4-m^2-\frac{3}{40}\right)A_5^2 \\
&\quad - 1960\left(m^2-\frac{1}{2}\right)^2A_3A_7\Big]
\end{aligned} \tag{62}$$

$$
\begin{aligned}
&\times A_4A_7\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}-150\left(48m^4-48m^2+9\right)A_5^3.\\
&\quad+\ 29,400\left(m^2-\frac{1}{2}\right)^2A_3A_5A_7\\
&\quad+144\left(m^4-m^2+\frac{5}{8}\right)A_4^3A_7^2\Bigg\},\\
&A_2=\frac{\left(-\frac{90}{A_7}\right)^{-\frac{2}{3}}}{(2m^2-1)^2A_7}\Bigg\{360\Bigg[\frac{2}{1125}A_4A_7\\
&\quad\left(m^4-m^2-\frac{37}{112}\right)\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\\
&\quad-\frac{1}{175}\left(m^4-m^2+1\right)^2A_5\Bigg]A_4A_7\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\\
&\quad+360\left[\left(m^2-\frac{1}{2}\right)^2A_3A_7\right.\\
&\quad\left.-\frac{11}{35}A_5^2\left(m^4-m^2+\frac{13}{88}\right)\right]\Bigg\},\\
&\quad\alpha_2=2\left(-\frac{720h_6^3}{A_7}\right)^{\frac{1}{6}},\alpha_1=0,\\
&\alpha_0=\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{\frac{(m^2-1)\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}+6A_4\right]}{210(2m^2-1)}},\\
&\quad h_4=2\sqrt{\frac{(m^2-1)\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}+6A_4\right]h_6}{210(2m^2-1)}},h_6=h_6,
\end{aligned}
\tag{62}
$$

provided

$$
(2m^2-1)\left[A_5\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}+6A_4\right]<0 \text{ and } A_7<0. \tag{63}
$$

Substituting (62) into Eq. (48), we have the Jacobi elliptic function solutions of Eq. (1) in the form:

$$
\begin{aligned}
&q(x,t)=\pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{-\frac{2(1-m^2)}{(2m^2-1)}\Delta}\\
&\quad\left[\mathrm{nc}\left(\frac{1}{2}\left[x-a_1+2a_2k+8k^3(a_4+2a_5k)\right]\sqrt{-\frac{2}{(2m^2-1)}\Delta}\right)\right]\\
&\quad\exp[i(-kx+\omega t+\theta],
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
&q(x,t)=\pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{-\frac{2}{(2m^2-1)}\Delta}\\
&\quad\left[\mathrm{ds}\left(\frac{1}{2}\left[x-a_1+2a_2k+8k^3(a_4+2a_5k)\right]\sqrt{-\frac{2}{(2m^2-1)}\Delta}\right)\right]\\
&\quad\exp[i(-kx+\omega t+\theta].
\end{aligned}
\tag{65}
$$

On using (11) and (62) one can show that the wave number ω in (64) and (65) is given by:

$$
\begin{aligned}
&\omega=\frac{1}{102900(2m^2-1)^2A_7^2k}\Big\{-392a_5A_4A_5A_7^2\\
&\quad\left[m^4\left(\frac{2}{49}A_4+k^2\right)\right.\\
&\quad\left.-m^2\left(\frac{2}{49}A_4+k^2\right)+\frac{3}{49}A_4+k^2\right]\left(-\frac{90}{A_7}\right)^{\frac{2}{3}}+68600A_7a_5\\
&\quad\times\Bigg\{\left(m^2-\frac{1}{2}\right)^2A_3A_7\left(\frac{1}{35}A_4+k^2\right)\\
&\quad-\frac{11}{35}A_5^2\left[m^4\left(k^2+\frac{15}{539}A_4\right)-m^2\left(k^2+\frac{15}{539}A_4\right)\right.\\
&\quad\left.+\frac{13}{88}k^2-\frac{9}{4312}A_4\right]\Bigg\}\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\\
&\quad-\ \Big[\left(144m^4-144m^2+90\right)A_4^3\\
&\quad+\ 10976k^2\left(m^4-m^2+\frac{37}{112}\right)A_4^2\\
&\quad\left.-2744000\left(m^2-\frac{1}{2}\right)^2k^6\right]a_5\\
&\quad+1234800\left(m^2-\frac{1}{2}\right)^2k^2\left(k^3a_4+\frac{1}{3}a_1\right)A_7^2\\
&\quad-29400A_3A_5A_7a_5\left(m^2-\frac{1}{2}\right)^2\\
&\quad-a_5A_5^3\left(7200m^2-1350-7200m^4\right)\Big\},
\end{aligned}
\tag{66}
$$

where

$$
a_5=\frac{1050(2m^2-1)^2k(a_2+6a_4k^2)}{\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}\left[4(m^4-m^2+1)A_4A_5\left(-\frac{90}{A_7}\right)^{\frac{1}{3}}-175(4m^4-4m^2+1)A_3+\frac{5A_5^2}{2A_7}(88m^4-88m^2+13)\right]-13125(2m^2-1)k^4+112\left(m^4-m^2+\frac{37}{112}\right)A_4^2}, \tag{67}
$$

or

in which $A_i(i=3,4,5,7)$ are given by (11).

In particular, if $m\to 1$, then $ds(\xi,1)\to$ csch (ξ). In this case, from (65) we find the singular soliton solutions of Eq. (1) in the form:

$$q(x,t) = \pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{-2\Delta}$$
$$\left[\operatorname{csch}\left(\frac{1}{2}\left[x - a_1 + 2a_2k + 8k^3(a_4 + 2a_5k)\right]\sqrt{-2\Delta}\right)\right]$$
$$\exp[i(-kx + \omega t + \theta]. \tag{68}$$

If $m \to 0$, then $nc(\xi, 0) \to \sec(\xi)$ and $ds(\xi, 0) \to \csc(\xi)$. In this case, from (64) and (65) we have the periodic solutions of Eq. (1) in the form:

$$q(x,t) = \pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{2\Delta}$$
$$\left[\sec\left(\frac{1}{2}\left[x - a_1 + 2a_2k + 8k^3(a_4 + 2a_5k)\right]\sqrt{2\Delta}\right)\right] \tag{69}$$
$$\exp[i(-kx + \omega t + \theta],$$

or

$$q(x,t) = \pm\frac{1}{2}\left(-\frac{720}{A_7}\right)^{\frac{1}{6}}\sqrt{2\Delta}$$
$$\left[\csc\left(\frac{1}{2}\left[x - a_1 + 2a_2k + 8k^3(a_4 + 2a_5k)\right]\sqrt{2\Delta}\right)\right] \tag{70}$$
$$\exp[i(-kx + \omega t + \theta].$$

7. Conclusions

This paper constructs several soliton solutions to highly dispersive NLSE with CQS nonlinearity. A number of integration algorithms have produced a wide spectrum of solutions possible. As a by-product of these integration schemes, several additional solutions such as periodic and Jacobi's elliptic function solutions also emerge. These are also listed, in order to gain a more complete spectrum of solutions to the model. These solutions serve as an encouragement to further proceed along with identifying highly dispersive solitons in other models. Immediate extension is to study models with birefringence and DWDM networks. Other aspects also need to be addressed in the present model. These include the demonstration of the conservation laws of the model and the integration of the model with the inclusion of perturbative terms.

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