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Solitons and conservation laws in magneto–optic waveguides with generalized Kudryashov's equation

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ABSTRACT

The modified sub–ODE approach secures optical soliton solutions in magneto–optic waveguides with generalized Kudryashov's equation. The solutions are initially drafted in terms of Jacobi's elliptic functions. The limiting process, when the modulus of ellipticity approaches zero or unity, the soliton solutions emerge. A few solutions in terms of Weierstrass' elliptic functions are also revealed. Finally, the conservation laws are computed for the model using the multiplier approach.

1. Introduction

An important application of optical solitons to a wide range of waveguides is in magneto–optic waveguides [1–20]. This carries several advantages such as controlling soliton cluster and thus transferring these soliton molecules to a state of "separation". This addresses a growing problem in the age of telecommunications, namely Internet bottleneck. Thus, it is imperative and vital to take a deeper look at soliton dynamics in such waveguides.

During 2019, Kudryashov proposed a new form of intensity–dependent refractive index that led to a wide range of reported results [9]. Later during 2020, this law was generalized and extended with eight nonlinear terms that gave way to what is being referred to as generalized Kudryashov's equation [10]. Today's paper implements this generalized Kudryashov's law of refractive index to magneto–optic waveguides and study the dynamics of solitons in it. Our approach in today's work is the sub–ODE scheme. This algorithm enables the governing model to yield bright, dark and singular soliton solutions to the model. The existence criteria of these solitons are

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also captured from the structure of the soliton solutions. Finally, the conservation laws and the conserved quantities are also recovered for the model. This gave way to the zone of existence for the solitons that depends on the power law nonlinearity parameter in Kudryashov's law of refractive index. The results along with the detailed analytical perspectives are all exhibited in the rest of the paper.

1.1. Governing model

The governing nonlinear Schrödinger's equation (NLSE) for the generalized Kudryashov's equation in polarization preserving fibers is written as [10]:

$$
\begin{aligned}
&iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + \sigma q \\
&+ \left(\frac{\nu_4}{|q|^{4n}} + \frac{\nu_3}{|q|^{3n}} + \frac{\nu_2}{|q|^{2n}} + \frac{\nu_1}{|q|^{n}} + \mu_1 |q|^n + \mu_2 |q|^{2n} + \mu_3 |q|^{3n} + \mu_4 |q|^{4n} \right) q = 0
\end{aligned} \tag{1}
$$

where $\sigma, a_m, \nu_m, \mu_m, (m = 1,2,3,4)$ are parameters, while $i = \sqrt{-1}$. The dependent variable $q = q(x,t)$ is the complex–valued describing the pulse profile. The independent variables x and t represent spatial and temporal variables, respectively. The nonlinearity index n is the power parameter.

The coupled system of NLSE for the generalized Kudryashov's equation in magneto-optic waveguides is written, for the first time, in the form:

$$
\begin{aligned}
&iu_t + ia_1 u_x + b_1 u_{xx} + ic_1 u_{xxx} + d_1 u_{xxxx} + \sigma_1 u \\
&+ \left(\frac{n_1}{|u|^{4n}} + \frac{m_1}{|u|^{3n}} + \frac{l_1}{|u|^{2n}} + \frac{k_1}{|u|^{n}} + e_1 |u|^n + f_1 |u|^{2n} + g_1 |u|^{3n} + h_1 |u|^{4n} \right) u \\
&+ \left(\frac{\tau_1}{|v|^{4n}} + \frac{\zeta_1}{|v|^{3n}} + \frac{\eta_1}{|v|^{2n}} + \frac{\xi_1}{|v|^{n}} + \alpha_1 |v|^n + \beta_1 |v|^{2n} + \gamma_1 |v|^{3n} + \delta_1 |v|^{4n} \right) u \\
&= Q_1 v + i\left[\lambda_1 \left(|u|^{2n} u \right)_x + \nu_1 \left(|u|^{2n} \right)_x u + \theta_1 |u|^{2n} u_x \right]
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
&iv_t + ia_2 v_x + b_2 v_{xx} + ic_2 v_{xxx} + d_2 v_{xxxx} + \sigma_2 v \\
&+ \left(\frac{}{n_2}|v|^{4n} + \frac{m_2}{|v|^{3n}} + \frac{l_2}{|v|^{2n}} + \frac{k_2}{|v|^{n}} + e_2 |v|^n + f_2 |v|^{2n} + g_2 |v|^{3n} + h_2 |v|^{4n} \right) v \\
&+ \left(\frac{\tau_2}{|u|^{4n}} + \frac{\zeta_2}{|u|^{3n}} + \frac{\eta_2}{|u|^{2n}} + \frac{\xi_2}{|u|^{n}} + \alpha_2 |u|^n + \beta_3 |u|^{2n} + \gamma_2 |u|^{3n} + \delta_2 |u|^{4n} \right) v \\
&= Q_2 u + i\left[\lambda_2 \left(|v|^{2n} v \right)_x + \nu_2 \left(|v|^{2n} \right)_x v + \theta_2 |v|^{2n} v_x \right]
\end{aligned} \tag{3}
$$

where $a_j, b_j, c_j, d_j, \sigma_j, e_j, f_j, g_j, h_j, k_j, l_j, m_j, n_j, \alpha_j, \beta_j, \gamma_j, \delta_j, \xi_j, \eta_j, \zeta_j, \tau_j, Q_j, \lambda_j, \nu_j$ and θ_j for $j = 1,2$ are all real constants, while $i = \sqrt{-1}$. The system (2) and (3) are coupled NLSEs for the generalization of Kudryashov's equation that govern the propagation of optical solitons through magneto–optic waveguides, where the independent variables x and t represent the spatial and temporal variables, respectively. The dependent variables $u(x,t)$ and $v(x,t)$ are the complex valued wave profiles, where a_j represent the coefficients of inter––modal dispersion (IMD), b_j are the coefficients of group velocity dispersion (GVD), c_j are the coefficients of third–order dispersion (3OD) and d_j are the coefficients of fourth–order dispersion (4OD). Next, $e_j, f_j, g_j, h_j, k_j, l_j, m_j$ and n_j are the coefficients of self-phase modulation (SPM). The parameters $\alpha_j, \beta_j, \gamma_j, \delta_j, \xi_j, \eta_j, \zeta_j$ and τ_j represent the cross-phase modulation (XPM). Here σ_j come from the detuning effect and Q_j are the coefficients of magneto-optic parameters. The parameters λ_j give the self-steepening (SS) terms that avoid the formation of shock waves. Finally ν_j and θ_j sand for the coefficients of nonlinear dispersion.

The objective of today's work is to reveal bright, dark and singular soliton solutions in magneto–optic waveguides with generalized Kudryashov's equation by the sub–ODE scheme and then compute the conservation laws for the model employing the multiplier approach. It worths noting that the results recovered are new and have not been reported before.

This article is organized as follows: In Section 2, mathematical analysis is discussed. In Section 3 we solve Eqs. (2) and (3) using the modified sub–ODE method. In Section 4, the conservation laws are displayed while finally in Section 5, conclusions are drawn.

2. Mathematical analysis

In order to solve Eqs. (2) and (3), we assume that the hypothesis as:

$$u(x,t) = \phi_1(\xi)\exp[i\psi(x,t)] \tag{4}$$

$$v(x,t) = \phi_2(\xi)\exp[i\psi(x,t)] \tag{5}$$

and

$$\xi = x - ct, \quad \psi(x,t) = -\kappa x + \omega t + \theta_0 \tag{6}$$

where c, κ, ω and θ_0 are all non zero constants to be determined which represent velocity of soliton, frequency of soliton, wave number and phase constant, respectively, while $\phi_1(\xi), \phi_2(\xi)$ and $\psi(x,t)$ are real functions which represent the amplitude portion of the soliton and the phase component of the soliton, respectively. Substituting (4) and (5) along with (6) into Eqs. (2) and (3), separating the real and imaginary parts, we deduce that the real parts are

$$\begin{aligned} &d_1\phi_1'''' + (b_1 + 3c_1\kappa - 6d_1\kappa^2)\phi_1'' + (\sigma_1 - \omega + a_1\kappa - b_1\kappa^2 - c_1\kappa^3 + d_1\kappa^4)\phi_1 - Q_1\phi_2 - \kappa(\lambda_1 + \theta_1)\phi_1^{2n+1} \\ &+e_1\phi_1^{1+n} + f_1\phi_1^{1+2n} + g_1\phi_1^{1+3n} + h_1\phi_1^{1+4n} + k_1\phi_1^{1-n} + l_1\phi_1^{1-2n} + m_1\phi_1^{1-3n} + n_1\phi_1^{1-4n} \\ &+(\alpha_1\phi_2^n + \beta_1\phi_2^{2n} + \gamma_1\phi_2^{3n} + \delta_1\phi_2^{4n} + \xi_1\phi_2^{-n} + \eta_1\phi_2^{-2n} + \zeta_1\phi_2^{-3n} + \tau_1\phi_2^{-4n})\phi_1 = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} &d_2\phi_2'''' + (b_2 + 3c_2\kappa - 6d_2\kappa^2)\phi_2'' + (\sigma_2 - \omega + a_2\kappa - b_2\kappa^2 - c_2\kappa^3 + d_2\kappa^4)\phi_2 - Q_2\phi_1 - \kappa(\lambda_2 + \theta_2)\phi_2^{2n+1} \\ &+e_2\phi_2^{1+n} + f_2\phi_2^{1+2n} + g_2\phi_2^{1+3n} + h_2\phi_2^{1+4n} + k_2\phi_2^{1-n} + l_2\phi_2^{1-2n} + m_2\phi_2^{1-3n} + n_2\phi_2^{1-4n} \\ &+(\alpha_2\phi_1^n + \beta_2\phi_1^{2n} + \gamma_2\phi_1^{3n} + \delta_2\phi_1^{4n} + \xi_2\phi_1^{-n} + \eta_2\phi_1^{-2n} + \zeta_2\phi_1^{-3n} + \tau_2\phi_1^{-4n})\phi_2 = 0 \end{aligned} \tag{8}$$

while the imaginary parts are

$$(c_1 - 4d_1\kappa)\phi_1''' + (a_1 - c - 2b_1\kappa - 3c_1\kappa^2 + 4d_1\kappa^3)\phi_1' - [(2n+1)\lambda_1 + 2n\nu_1 + \theta_1]\phi_1^{2n}\phi_1' = 0 \tag{9}$$

$$(c_2 - 4d_2\kappa)\phi_2''' + (a_2 - c - 2b_2\kappa - 3c_2\kappa^2 + 4d_2\kappa^3)\phi_2' - [(2n+1)\lambda_2 + 2n\nu_2 + \theta_2]\phi_2^{2n}\phi_2' = 0. \tag{10}$$

The linearly independent principle is applied on (9) and (10) to get:

$$c_1 - 4d_1\kappa = 0 \tag{11}$$

$$c = a_1 - 2b_1\kappa - 8d_1\kappa^3 \tag{12}$$

$$(2n+1)\lambda_1 + 2n\nu_1 + \theta_1 = 0 \tag{13}$$

and

$$c_2 - 4d_2\kappa = 0 \tag{14}$$

$$c = a_2 - 2b_2\kappa - 8d_2\kappa^3 \tag{15}$$

$$(2n+1)\lambda_2 + 2n\nu_2 + \theta_2 = 0. \tag{16}$$

From (12) and (15), one gets the condition:

$$a_1 - a_2 - 2\kappa(b_1 - b_2) - 8\kappa^3(d_1 - d_2) = 0. \tag{17}$$

From (11) and (14), we deduce that the frequency of soliton κ is given by:

$$\kappa = \frac{c_j}{4d_j} \tag{18}$$

provided $d_j \neq 0$ for $j = 1, 2$. From (18) we have the relation

$$c_1 d_2 = c_2 d_1. \tag{19}$$

Set

$$\phi_2(\xi) = \chi\phi_1(\xi) \tag{20}$$

where χ is a nonzero constant, such that $\chi \neq 1$. Consequently, Eqs. (7) and (8) reduce to

$$\begin{aligned}&d_1\phi_1'''' + \left(b_1 + 3c_1\kappa - 6d_1\kappa^2\right)\phi_1'' + \left(\sigma_1 - \omega + a_1\kappa - b_1\kappa^2 - c_1\kappa^3 + d_1\kappa^4 - Q_1\chi\right)\phi_1\\&+(e_1 + \alpha_1\chi^n)\phi_1^{1+n} + \left[f_1 + \beta_1\chi^{2n} - \kappa(\lambda_1 + \theta_1)\right]\phi_1^{1+2n} + \left(g_1 + \gamma_1\chi^{3n}\right)\phi_1^{1+3n}\\&+(h_1 + \delta_1\chi^{4n})\phi_1^{1+4n} + (k_1 + \xi_1\chi^{-n})\phi_1^{1-n} + \left(l_1 + \eta_1\chi^{-2n}\right)\phi_1^{1-2n}\\&+(m_1 + \zeta_1\chi^{-3n})\phi_1^{1-3n} + \left(n_1 + \tau_1\chi^{-4n}\right)\phi_1^{1-4n} = 0\end{aligned} \tag{21}$$

and

$$\begin{aligned}&d_2\chi\phi_1'''' + \left(b_2 + 3c_2\kappa - 6d_2\kappa^2\right)\chi\phi_1'' + \left[\left(\sigma_2 - \omega + a_2\kappa - b_2\kappa^2 - c_2\kappa^3 + d_2\kappa^4\right)\chi - Q_2\right]\phi_1\\&+(e_2\chi^{1+n} + \alpha_2\chi)\phi_1^{1+n} + \left[f_2\chi^{1+2n} + \beta_2\chi - \kappa(\lambda_2 + \theta_2)\chi^{2n+1}\right]\phi_1^{2n+1} + \left(g_2\chi^{1+3n} + \gamma_2\chi\right)\phi_1^{1+3n}\\&+(h_2\chi^{1+4n} + \delta_2\chi)\phi_1^{1+4n} + \left(k_2\chi^{1-n} + \xi_2\chi\right)\phi_1^{1-n} + \left(l_2\chi^{1-2n} + \eta_1\chi\right)\phi_1^{1-2n}\\&+(m_2\chi^{1-3n} + \zeta_2\chi)\phi_1^{1-3n} + \left(n_2\chi^{1-4n} + \tau_2\chi\right)\phi_1^{1-4n} = 0.\end{aligned} \tag{22}$$

Eqs. (21) and (2) have the same form under the constraint conditions:

$$d_1 = d_2\chi \tag{23}$$

$$b_1 + 3c_1\kappa - 6d_1\kappa^2 = \left(b_2 + 3c_2\kappa - 6d_2\kappa^2\right)\chi \tag{24}$$

$$\sigma_1 - \omega + a_1\kappa - b_1\kappa^2 - c_1\kappa^3 + d_1\kappa^4 - Q_1\chi = \left(\sigma_2 - \omega + a_2\kappa - b_2\kappa^2 - c_2\kappa^3 + d_2\kappa^4\right)\chi - Q_2 \tag{25}$$

$$e_1 + \alpha_1\chi^n = e_2\chi^{1+n} + \alpha_2\chi \tag{26}$$

$$f_1 + \beta_1\chi^{2n} - \kappa(\lambda_1 + \theta_1) = f_2\chi^{1+2n} + \beta_2\chi - \kappa(\lambda_2 + \theta_2)\chi^{2n+1} \tag{27}$$

$$g_1 + \gamma_1\chi^{3n} = g_2\chi^{1+3n} + \gamma_2\chi \tag{28}$$

$$h_1 + \delta_1\chi^{4n} = h_2\chi^{1+4n} + \delta_2\chi \tag{29}$$

$$k_1 + \xi_1\chi^{-n} = k_2\chi^{1-n} + \xi_2\chi \tag{30}$$

$$l_1 + \eta_1\chi^{-2n} = l_2\chi^{1-2n} + \eta_1\chi \tag{31}$$

$$m_1 + \zeta_1\chi^{-3n} = m_2\chi^{1-3n} + \zeta_2\chi \tag{32}$$

$$n_1 + \tau_1\chi^{-4n} = n_2\chi^{1-4n} + \tau_2\chi. \tag{33}$$

From (19), (23) and (25), the wave number ω is given by

$$\omega = \frac{Q_2 - Q_1\chi - (\chi a_2 - a_1)\kappa + (\chi b_2 - b_1)\kappa^2 - \chi\sigma_2 + \sigma_1}{(1-\chi)}. \tag{34}$$

Eq. (21) can be rewritten in the form:

$$\begin{aligned}&\phi_1'''' + H_0\phi_1'' + H_1\phi_1 + \Delta_1\phi_1^{1+n} + \Delta_2\phi_1^{1+2n} + \Delta_3\phi_1^{1+3n} + \Delta_4\phi_1^{1+4n}\\&+\Gamma_1\phi_1^{1-n} + \Gamma_2\phi_1^{1-2n} + \Gamma_3\phi_1^{1-3n} + \Gamma_4\phi_1^{1-4n} = 0\end{aligned} \tag{35}$$

where

$$\left.\begin{aligned}&H_0 = \frac{b_1 + 3c_1\kappa - 6d_1\kappa^2}{d_1}, \quad H_1 = \frac{\sigma_1 - \omega + a_1\kappa - b_1\kappa^2 - c_1\kappa^3 + d_1\kappa^4 - Q_1\chi}{d_1},\\&\Delta_1 = \frac{e_1 + \alpha_1\chi^n}{d_1}, \quad \Delta_2 = \frac{f_1 + \beta_1\chi^{2n} - \kappa(\lambda_1 + \theta_1)}{d_1}, \quad \Delta_3 = \frac{g_1 + \gamma_1\chi^{3n}}{d_1}, \quad \Delta_4 = \frac{h_1 + \delta_1\chi^{4n}}{d_1},\\&\Gamma_1 = \frac{k_1 + \xi_1\chi^{-n}}{d_1}, \quad \Gamma_2 = \frac{l_1 + \eta_1\chi^{-2n}}{d_1}, \quad \Gamma_3 = \frac{m_1 + \zeta_1\chi^{-3n}}{d_1}, \quad \Gamma_4 = \frac{n_1 + \tau_1\chi^{-4n}}{d_1}.\end{aligned}\right\} \tag{36}$$

Now, the problem of finding exact solutions for Eqs. (2) and (3) is transformed for solving Eq. (2).

3. The modified sub-ode method

Let us demonstrate that there is a solution of Eq. (2) in the form

$$\phi_1(\xi) = F(\xi) \tag{37}$$

where $F(\xi)$ is the solution of the following modified sub–ODE equation [20–25]:

$$F'^2(\xi) = AF^{2-2n}(\xi) + BF^{2-n}(\xi) + CF^2(\xi) + DF^{2+n}(\xi) + EF^{2+2n}(\xi), \quad n > 0 \tag{38}$$

where A, B, C, D and E are constants. We will use (37) and (38) to get

$$\phi_1''(\xi) = (1-n)AF^{1-2n}(\xi) + \frac{1}{2}(2-n)BF^{1-n}(\xi) + CF(\xi) + \frac{1}{2}(2+n)DF^{1+n}(\xi) + (1+n)EF^{1+2n}(\xi) \tag{39}$$

$$\begin{aligned}\phi_1'''(\xi) = &\Big[(1-n)(1-2n)AF^{-2n}(\xi) + \frac{1}{2}(2-n)(1-n)BF^{-n}(\xi) + C \\ &+\frac{1}{2}(2+n)(1+n)DF^{n}(\xi) + (1+n)(1+2n)EF^{2n}(\xi)\Big]F_1'(\xi)\end{aligned} \tag{40}$$

and

$$\begin{aligned}\phi_1''''(\xi) = &(1-n)\left[\frac{n^2}{4}\left(3B^2+16AC\right) + (1-2n)\left(2AC+B^2\right)\right]F^{1-2n}(\xi) \\ &+(2-n)\left[\frac{n^2}{2}(BC+6AD) + (1-n)(BC+AD)\right]F^{1-n}(\xi) \\ &+(1+n)\left[\frac{n^2}{4}\left(3D^2+16EC\right) + (2n+1)\left(D^2+2EC\right)\right]F^{1+2n}(\xi) \\ &+(3n-1)(2n-1)(1-n)A^2F^{1-4n}(\xi) + (3n-2)(2n-1)(1-n)ABF^{1-3n}(\xi) \\ &+(3n+2)(2n+1)(1+n)EDF^{1+3n}(\xi) + (2n+1))(3n+1)(1+n)E^2F^{1+4n}(\xi) \\ &+(2+n)\left[\frac{n^2}{2}(DC+6BE) + (DC+BE)(n+1)\right]F^{1+n}(\xi) \\ &+\left[\frac{5n^2}{2}(4AE+DB) + 2(AE+DB) + C^2\right]F(\xi).\end{aligned} \tag{41}$$

Substituting (37), (39) and (41) into Eq. (2), collecting all the coefficients of $[F(\xi)]^{Jn+1}[F'(\xi)]^s$, $(J = 0, 1, 2, \ldots, 8, s = 0, 1)$ and setting them to zero, one gets the following algebraic equations:

$$\left.\begin{aligned}&(3n+1)(2n+1)(1+n)E^2 + \Delta_4 = 0, \\ &(3n+2)(2n+1)(1+n)ED + \Delta_3 = 0, \\ &(1+n)\left[n^2\left(\frac{3}{4}D^2+4EC\right) + (2n+1)\left(D^2+2EC\right)\right] + (1+n)H_0E + \Delta_2 = 0, \\ &(2+n)\left[(6BE+DC)n^2 + 2(n+1)(BE+DC)\right] + (n+2)DH_0 + 2\Delta_1 = 0, \\ &\frac{5n^2}{2}(4AE+DB) + 2(DB+AE) + C^2 + H_0C + H_1 = 0, \\ &(n-2)\left[(6AD+BC)n^2 + 2(1-n)(BC+AD)\right] + (n-2)H_0B - 2\Gamma_1 = 0, \\ &(1-n)\left[\left(\frac{3}{4}B^2+4AC\right)n^2 + (1-2n)\left(2AC+B^2\right)\right] + \Gamma_2 + (1-n)AH_0 = 0, \\ &(2n-1)(3n-2)(1-n)BA + \Gamma_3 = 0, \\ &(2n-1)(3n-1)(1-n)A^2 + \Gamma_4 = 0.\end{aligned}\right\} \tag{42}$$

Solving the above algebraic Eq. (42), one gets the following conditions for existence of the solutions $\phi_1(\xi) = F(\xi)$ of Eq. (2):

$$\Delta_1 = -(2+n)\left[\left(3BE + \frac{1}{2}DC\right)n^2 + (n+1)(BE+DC) + \frac{1}{2}DH_0\right] \tag{43}$$

$$\Delta_2 = -(1+n)\left[n^2\left(\frac{3}{4}D^2 + 4EC\right) + (2n+1)\left(D^2+2EC\right) + H_0E\right] \tag{44}$$

$$\Delta_3 = -(3n+2)(2n+1)(1+n)ED \tag{45}$$

$$\Delta_4 = -(3n+1)(2n+1)(1+n)E^2 \tag{46}$$

$$H_1 = -5n^2\left(2AE + \frac{1}{2}DB\right) - 2(DB + AE) - C^2 - H_0C \tag{47}$$

$$\Gamma_1 = \frac{(n-2)}{2}\left[(6AD + BC)n^2 + 2(1-n)(BC + AD) + H_0B\right] \tag{48}$$

$$\Gamma_2 = (n-1)\left[\left(\frac{3}{4}B^2 + 4AC\right)n^2 + (1-2n)\left(2AC + B^2\right) + AH_0\right] \tag{49}$$

$$\Gamma_3 = (2n-1)(3n-2)(n-1)BA \tag{50}$$

$$\Gamma_4 = (2n-1)(3n-1)(n-1)A^2. \tag{51}$$

According to the modified Sub-ODE method, we have the following types of solutions for Eqs. (2) and (3):

Case–1: If we substitute $A = B = D = 0$ in the above conditions (43)–(51), then one gets the results:

$$\begin{aligned} &H_1 = -C(C + H_0), \quad \Delta_1 = 0, \quad \Delta_2 = -(1+n)E\left[\left(4n^2 + 4n + 2\right)C + H_0\right], \quad \Delta_3 = 0, \\ &\Delta_4 = -(3n+1)(2n+1)(1+n)E^2, \quad \Gamma_1 = 0, \quad \Gamma_2 = 0, \quad \Gamma_3 = 0, \quad \Gamma_4 = 0. \end{aligned} \tag{52}$$

Under the constraint conditions (52), we have the solutions:

(I) The bright soliton solutions:

$$u(x,t) = \left[\varepsilon\sqrt{-\frac{C}{E}}\operatorname{sech}\left(n\sqrt{C}\,(x-ct)\right)\right]^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta_0)} \tag{53}$$

$$v(x,t) = \chi\left[\varepsilon\sqrt{-\frac{C}{E}}\operatorname{sech}\left(n\sqrt{C}\,(x-ct)\right)\right]^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta_0)} \tag{54}$$

provided $\varepsilon = \pm 1$, $C > 0$ and $E < 0$.

(II) The periodic solutions:

$$u(x,t) = \left[\varepsilon\sqrt{-\frac{C}{E}}\sec\left(n\sqrt{-C}\,(x-ct)\right)\right]^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta_0)} \tag{55}$$

$$v(x,t) = \chi\left[\varepsilon\sqrt{-\frac{C}{E}}\sec\left(n\sqrt{-C}\,(x-ct)\right)\right]^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta_0)} \tag{56}$$

provided $\varepsilon = \pm 1$, $C < 0$ and $E > 0$.

(III) The rational solutions:

$$u(x,t) = \left[\frac{\varepsilon}{n\sqrt{E}(x-ct)}\right]^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta_0)} \tag{57}$$

$$v(x,t) = \chi\left[\frac{\varepsilon}{n\sqrt{E}(x-ct)}\right]^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta_0)} \tag{58}$$

provided $\varepsilon = \pm 1$, $C = 0$ and $E > 0$.

Case–2: If we substitute $B = D = 0, A = \frac{C^2}{4E}$, in the above conditions (43)–(51), then one gets the results:

$$\begin{aligned} &H_1 = -\left[\frac{1}{2}\left(5n^2 + 3\right)C + H_0\right]C, \quad \Delta_1 = 0, \quad \Delta_2 = -(1+n)E\left[\left(4n^2 + 4n + 2\right)C + H_0\right], \quad \Delta_3 = 0, \\ &\Delta_4 = -(3n+1)(2n+1)(1+n)E^2, \quad \Gamma_1 = 0, \quad \Gamma_2 = \frac{(n-1)C^2}{4E}\left[C\left(4n^2 - 4n + 2\right) + H_0\right], \\ &\Gamma_3 = 0, \quad \Gamma_4 = \frac{C^4}{16E^2}(2n-1)(3n-1)(n-1). \end{aligned} \tag{59}$$

Under the constraint conditions (59), we have the solutions:

(I) The dark soliton solutions:

$$u(x,t)=\left[\varepsilon\sqrt{-\frac{C}{2E}}\tanh\left(n\sqrt{-\frac{C}{2}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{60}$$

$$v(x,t)=\chi\left[\varepsilon\sqrt{-\frac{C}{2E}}\tanh\left(n\sqrt{-\frac{C}{2}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{61}$$

provided $\varepsilon=\pm1$, $C<0$ and $E>0$.

(II) The periodic solutions:

$$u(x,t)=\left[\varepsilon\sqrt{\frac{C}{2E}}\tan\left(n\sqrt{\frac{C}{2}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{62}$$

$$v(x,t)=\chi\left[\varepsilon\sqrt{\frac{C}{2E}}\tan\left(n\sqrt{\frac{C}{2}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{63}$$

provided $\varepsilon=\pm1$, $C>0$ and $E>0$.

Case–3: If we substitute $B=D=0$, in the above conditions (43)–(51), then, one arrives at the results:

$$\begin{aligned}&H_1=-\left[2\left(5n^2+1\right)EA+(C+H_0)C\right],\quad \Delta_1=0,\quad \Delta_2=-(1+n)E\left[2C\left(2n^2+2n+1\right)+H_0\right],\\&\Delta_3=0,\quad \Delta_4=-(3n+1)(2n+1)(1+n)E^2,\quad \Gamma_2=A(n-1)\left[2C\left(2n^2-2n+1\right)+H_0\right],\\&\Gamma_1=0,\quad \Gamma_3=0,\quad \Gamma_4=(2n-1)(3n-1)(n-1)A^2.\end{aligned} \tag{64}$$

Under the constraint conditions (64), we have the solutions:

(I) If $A=\frac{m^2(m^2-1)C^2}{E(2m^2-1)^2}$ where $0<m<1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t)=\left[\varepsilon\sqrt{-\frac{Cm^2}{E(2m^2-1)}}\mathrm{cn}\left(n\sqrt{\frac{C}{2m^2-1}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{65}$$

$$v(x,t)=\chi\left[\varepsilon\sqrt{-\frac{Cm^2}{E(2m^2-1)}}\mathrm{cn}\left(n\sqrt{\frac{C}{2m^2-1}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{66}$$

provided

$$\varepsilon=\pm1,\ C\left(2m^2-1\right)>0 \text{ and } E<0. \tag{67}$$

(II) If $A=\frac{C^2(1-m^2)}{E(2-m^2)^2}$ where $0<m<1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t)=\left[\varepsilon\sqrt{-\frac{C}{E(2-m^2)}}\mathrm{dn}\left(n\sqrt{\frac{C}{2-m^2}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{68}$$

$$v(x,t)=\chi\left[\varepsilon\sqrt{-\frac{C}{E(2-m^2)}}\mathrm{dn}\left(n\sqrt{\frac{C}{2-m^2}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{69}$$

provided $\varepsilon=\pm1$, $C>0$ and $E<0$.

(III) If $A=\frac{C^2m^2}{E(m^2+1)^2}$ where $0<m<1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t)=\left[\varepsilon\sqrt{-\frac{Cm^2}{E(m^2+1)}}\mathrm{sn}\left(n\sqrt{-\frac{C}{m^2+1}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{70}$$

$$v(x,t)=\chi\left[\varepsilon\sqrt{-\frac{Cm^2}{E(m^2+1)}}\mathrm{sn}\left(n\sqrt{-\frac{C}{m^2+1}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{71}$$

provided $\varepsilon=\pm1$, $C<0$ and $E>0$.

(IV) The Weierstrass elliptic function solutions

$$u(x,t)=\left[\frac{1}{E}\wp(n(x-ct),g_2,g_3)-\frac{C}{3E}\right]^{\frac{1}{2n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{72}$$

$$v(x,t)=\chi\left[\frac{1}{E}\wp(n(x-ct),g_2,g_3)-\frac{C}{3E}\right]^{\frac{1}{2n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{73}$$

or

$$u(x,t)=\left[\frac{3A}{3\wp(n(x-ct),g_2,g_3)-C}\right]^{\frac{1}{2n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{74}$$

$$v(x,t)=\chi\left[\frac{3A}{3\wp(n(x-ct),g_2,g_3)-C}\right]^{\frac{1}{2n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{75}$$

where

$$g_2=\frac{4C^2-12AE}{3}\quad\text{and}\quad g_3=\frac{4C\left(-2C^2+9AE\right)}{27}. \tag{76}$$

Here $\wp(\xi,g_2,g_3)$ is called a Weierstrass elliptic function which satisfies the Eq. $\wp'^2=4\wp^3-g_2\wp-g_3$, such that g_2 and g_3 are called invariants of the Weierstrass elliptic function, in which $'=\frac{d}{d\xi}$.

(V) The Weierstrass elliptic function solutions

$$u(x,t)=\left[\frac{6\sqrt{A}\wp(n(x-ct),g_2,g_3)+C\sqrt{A}}{3\wp'(n(x-ct),g_2,g_3)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{77}$$

$$v(x,t)=\chi\left[\frac{6\sqrt{A}\wp(n(x-ct),g_2,g_3)+C\sqrt{A}}{3\wp'(n(x-ct),g_2,g_3)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{78}$$

or

$$u(x,t)=\left[\frac{3\sqrt{E^{-1}}\wp'(n(x-ct),g_2,g_3)}{6\wp(n(x-ct),g_2,g_3)+C}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{79}$$

$$v(x,t)=\chi\left[\frac{3\sqrt{E^{-1}}\wp'(n(x-ct),g_2,g_3)}{6\wp(n(x-ct),g_2,g_3)+C}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{80}$$

provided $A>0$ and $E>0$, where

$$g_2=\frac{C^2}{12}+AE\quad\text{and}\quad g_3=\frac{C\left(36AE-C^2\right)}{216}. \tag{81}$$

Case–4: If we substitute $A=B=E=0$, in the above conditions (43)–(51), then, one gets the results:

$$\begin{aligned}&H_1=-C(C+H_0),\quad \Delta_1=-\frac{(2+n)D}{2}\left[C\left(n^2+2n+2\right)+H_0\right],\\&\Delta_2=-\frac{1}{4}(1+n)\left(3n^2+8n+4\right)D^2,\quad \Delta_3=0,\quad \Delta_4=0,\\&\Gamma_1=0,\quad \Gamma_2=0,\quad \Gamma_3=0,\quad \Gamma_4=0.\end{aligned} \tag{82}$$

Under the constraint conditions (82), we have the solutions:

(I) The bright soliton solutions:

$$u(x,t)=\left[-\frac{C}{D}\text{sech}^2\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{83}$$

$$v(x,t)=\chi\left[-\frac{C}{D}\text{sech}^2\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{84}$$

provided $C>0$ and $D<0$.

(II) The periodic solutions:

$$u(x,t)=\left[-\frac{C}{D}\sec^2\left(\frac{n}{2}\sqrt{-C}\,(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{85}$$

$$v(x,t)=\chi\left[-\frac{C}{D}\sec^2\left(\frac{n}{2}\sqrt{-C}\,(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{86}$$

provided $C<0$ and $D>0$.

(III) The rational solutions:

$$u(x,t)=\left[\frac{4}{n^2D(x-ct)^2}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{87}$$

$$v(x,t)=\chi\left[\frac{4}{n^2D(x-ct)^2}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{88}$$

provided $C=0$ and $D>0$. **Case–5**: If we substitute $C=E=0$, in the above conditions (43)–(51), then, one gets the results:

$$\begin{aligned}
&H_1=-\frac{1}{2}\left(5n^2+4\right)DB,\quad \Delta_1=-\frac{1}{2}(2+n)DH_0,\\
&\Delta_2=-\frac{1}{4}D^2(1+n)\left(3n^2+8n+4\right),\quad \Delta_3=0,\quad \Delta_4=0,\\
&\Gamma_1=\frac{(n-2)}{2}\left[\left(6n^2-2n+2\right)AD+H_0B\right],\quad \Gamma_2=\frac{(n-1)}{4}\left[B^2(3n-2)(n-2)+4AH_0\right],\\
&\Gamma_3=(2n-1)(3n-2)(n-1)BA,\quad \Gamma_4=(2n-1)(3n-1)(n-1)A^2.
\end{aligned} \tag{89}$$

Under the constraint conditions (89), we have the solutions:

$$u(x,t)=\left[\wp\left(\frac{n}{2}\sqrt{D}\,(x-ct),-\frac{4B}{D},-\frac{4A}{D}\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{90}$$

$$v(x,t)=\chi\left[\wp\left(\frac{n}{2}\sqrt{D}\,(x-ct),-\frac{4B}{D},-\frac{4A}{D}\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{91}$$

provided $D>0$.

Case–6: If we substitute $A=B=0$ in the above conditions (43)–(51), then one gets the results:

$$\begin{aligned}
&H_1=-C(C+H_0),\quad \Delta_1=-\frac{(2+n)}{2}D\left[C\left(n^2+2n+2\right)+H_0\right],\\
&\Delta_2=-(1+n)\left\{\left[n^2\left(\frac{3}{4}D^2+4EC\right)+(2n+1)\left(D^2+2EC\right)\right]+H_0E\right\},\\
&\Delta_3=-(3n+2)(2n+1)(1+n)ED,\quad \Delta_4=-(3n+1)(2n+1)(1+n)E^2,\\
&\Gamma_1=0,\quad \Gamma_2=0,\quad \Gamma_3=0,\quad \Gamma_4=0.
\end{aligned} \tag{92}$$

Under the constraint conditions (92), we have the solutions:

(I) If $E=\frac{D^2}{4C}-C$ and $C>0$, then one gets the bright soliton solutions:

$$u(x,t)=\left[\frac{1}{\cosh\left(n\sqrt{C}(x-ct)\right)-\frac{D}{2C}}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{93}$$

$$v(x,t)=\chi\left[\frac{1}{\cosh\left(n\sqrt{C}(x-ct)\right)-\frac{D}{2C}}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{94}$$

provided $D<2C$.

(II) If $D=-2\sqrt{CE}$ and $C>0$, then one gets the dark soliton solutions:

$$u(x,t)=\left[\frac{1}{2}\sqrt{\frac{C}{E}}\left[1+\varepsilon\tanh\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right]\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{95}$$

$$v(x,t)=\chi\left[\frac{1}{2}\sqrt{\frac{C}{E}}\left[1+\varepsilon\tanh\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right]\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{96}$$

provided $\varepsilon=\pm1$ and $E>0$.

(III) If $C=0$, then one gets the rational solutions:

$$u(x,t)=\left[\frac{4D}{[nD(x-ct)]^2-4E}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{97}$$

$$v(x,t)=\chi\left[\frac{4D}{[nD(x-ct)]^2-4E}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{98}$$

provided $E<0$ and $D>0$.

(IV) If $D^2-4CE>0$ and $C>0$, then one gets the bright soliton solutions:

$$u(x,t)=\left[\frac{2C\operatorname{sech}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{2\sqrt{D^2-4CE}-\left[\sqrt{D^2-4CE}+D\right]\operatorname{sech}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{99}$$

$$v(x,t)=\chi\left[\frac{2C\operatorname{sech}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{2\sqrt{D^2-4CE}-\left[\sqrt{D^2-4CE}+D\right]\operatorname{sech}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{100}$$

or

$$u(x,t)=\left[\frac{2C}{\varepsilon\sqrt{D^2-4CE}\cosh\left(n\sqrt{C}(x-ct)\right)-D}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{101}$$

$$v(x,t)=\chi\left[\frac{2C}{\varepsilon\sqrt{D^2-4CE}\cosh\left(n\sqrt{C}(x-ct)\right)-D}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{102}$$

and the singular soliton solutions:

$$u(x,t)=\left[\frac{2C\operatorname{csch}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{2\sqrt{D^2-4CE}+\left[\sqrt{D^2-4CE}-D\right]\operatorname{csch}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{103}$$

$$v(x,t)=\chi\left[\frac{2C\operatorname{csch}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{2\sqrt{D^2-4CE}+\left[\sqrt{D^2-4CE}-D\right]\operatorname{csch}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{104}$$

provided $\varepsilon=\pm1$.

(V) If $D^2-4CE<0$ and $C>0$, then one gets the singular soliton solutions:

$$u(x,t)=\left[\frac{2C}{\varepsilon\sqrt{-\left(D^2-4CE\right)}\sinh\left(n\sqrt{C}(x-ct)\right)-D}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{105}$$

$$v(x,t)=\chi\left[\frac{2C}{\varepsilon\sqrt{-\left(D^2-4CE\right)}\sinh\left(n\sqrt{C}(x-ct)\right)-D}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{106}$$

provided $\varepsilon=\pm1$.

(VI) If $D^2-4CE=0$ and $C>0$, then one gets the bright soliton solutions:

$$u(x,t)=\left[-\frac{C}{D}\left\{1+\varepsilon\tanh\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{107}$$

$$v(x,t)=\chi\left[-\frac{C}{D}\left\{1+\varepsilon\tanh\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{108}$$

and the singular soliton solutions:

$$u(x,t)=\left[-\frac{C}{D}\left\{1+\varepsilon\coth\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{109}$$

$$v(x,t)=\chi\left[-\frac{C}{D}\left\{1+\varepsilon\coth\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{110}$$

provided $\varepsilon=\pm 1$.

(VII) If $C>0$, one gets the solitary wave solutions:

$$u(x,t)=\left[-\frac{CD\operatorname{sech}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{D^2-CE\left[1+\varepsilon\tanh\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right]^2}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{111}$$

$$v(x,t)=\chi\left[-\frac{CD\operatorname{sech}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{D^2-CE\left[1+\varepsilon\tanh\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right]^2}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{112}$$

or

$$u(x,t)=\left[\frac{CD\operatorname{csch}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{D^2-CE\left[1+\varepsilon\coth\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right]^2}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{113}$$

$$v(x,t)=\chi\left[\frac{CD\operatorname{csch}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{D^2-CE\left[1+\varepsilon\coth\left(\frac{n}{2}\sqrt{C}\,(x-ct)\right)\right]^2}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{114}$$

or

$$u(x,t)=\left[-\frac{C\operatorname{sech}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{D+2\varepsilon\sqrt{CE}\tanh\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{115}$$

$$v(x,t)=\chi\left[-\frac{C\operatorname{sech}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{D+2\varepsilon\sqrt{CE}\tanh\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{116}$$

or

$$u(x,t)=\left[\frac{C\operatorname{csch}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{D+2\varepsilon\sqrt{CE}\coth\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{117}$$

$$v(x,t)=\chi\left[\frac{C\operatorname{csch}^2\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}{D+2\varepsilon\sqrt{CE}\coth\left(\frac{n}{2}\sqrt{C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{118}$$

provided $\varepsilon=\pm 1$ and $E>0$.

(VIII) If $D^2-4CE>0$ and $C<0$, then one gets the periodic solutions:

$$v(x,t)=\left[\frac{-2C\sec^2\left(\frac{\sqrt{-C}}{2}n(x-ct)\right)}{2\sqrt{D^2-4CE}-\left[\sqrt{D^2-4CE}-D\right]\sec^2\left(\frac{\sqrt{-C}}{2}n(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{119}$$

$$v(x,t)=\chi\left[\frac{-2C\sec^2\left(\frac{\sqrt{-C}}{2}n(x-ct)\right)}{2\sqrt{D^2-4CE}-\left[\sqrt{D^2-4CE}-D\right]\sec^2\left(\frac{\sqrt{-C}}{2}n(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{120}$$

or

$$u(x,t)=\left[\frac{2C\csc^2\left(\frac{\sqrt{-C}}{2}n(x-ct)\right)}{2\sqrt{D^2-4CE}-\left[\sqrt{D^2-4CE}+D\right]\csc^2\left(\frac{\sqrt{-C}}{2}n(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{121}$$

$$v(x,t)=\chi\left[\frac{2C\csc^2\left(\frac{\sqrt{-C}}{2}n(x-ct)\right)}{2\sqrt{D^2-4CE}-\left[\sqrt{D^2-4CE}+D\right]\csc^2\left(\frac{\sqrt{-C}}{2}n(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{122}$$

or

$$u(x,t)=\left[\frac{2C\sec\left(n\sqrt{-C}(x-ct)\right)}{\varepsilon\sqrt{D^2-4CE}-D\sec\left(n\sqrt{-C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{123}$$

$$v(x,t)=\chi\left[\frac{2C\sec\left(n\sqrt{-C}(x-ct)\right)}{\varepsilon\sqrt{D^2-4CE}-D\sec\left(n\sqrt{-C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{124}$$

or

$$u(x,t)=\left[\frac{2C\csc\left(n\sqrt{-C}(x-ct)\right)}{\varepsilon\sqrt{D^2-4CE}-D\csc\left(n\sqrt{-C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{125}$$

$$v(x,t)=\chi\left[\frac{2C\csc\left(n\sqrt{-C}(x-ct)\right)}{\varepsilon\sqrt{D^2-4CE}-D\csc\left(n\sqrt{-C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{126}$$

provided $\varepsilon=\pm1$.

(IX) If $C<0$, then one gets the periodic solutions:

$$u(x,t)=\left[-\frac{C\sec^2\left(\frac{n}{2}\sqrt{-C}(x-ct)\right)}{D+2\varepsilon\sqrt{-CE}\tan\left(\frac{n}{2}\sqrt{-C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{127}$$

$$v(x,t)=\chi\left[-\frac{C\sec^2\left(\frac{n}{2}\sqrt{-C}(x-ct)\right)}{D+2\varepsilon\sqrt{-CE}\tan\left(\frac{n}{2}\sqrt{-C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{128}$$

or

$$u(x,t)=\left[-\frac{C\csc^2\left(\frac{n}{2}\sqrt{-C}(x-ct)\right)}{D+2\varepsilon\sqrt{-CE}\cot\left(\frac{n}{2}\sqrt{-C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{129}$$

$$v(x,t)=\chi\left[-\frac{C\csc^2\left(\frac{n}{2}\sqrt{-C}(x-ct)\right)}{D+2\varepsilon\sqrt{-CE}\cot\left(\frac{n}{2}\sqrt{-C}(x-ct)\right)}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{130}$$

provided $\varepsilon = \pm 1$ and $E > 0$.

(X) If $C > 0$, then one gets the solutions:

$$u(x,t) = \left[\frac{4Cn^2 e^{\varepsilon n\sqrt{C}(x-ct)}}{\left(e^{\varepsilon n\sqrt{C}(x-ct)} - Dn^2\right)^2 - 4CEn^4}\right]^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta_0)} \tag{131}$$

$$v(x,t) = \chi\left[\frac{4Cn^2 e^{\varepsilon n\sqrt{C}(x-ct)}}{\left(e^{\varepsilon n\sqrt{C}(x-ct)} - Dn^2\right)^2 - 4CEn^4}\right]^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta_0)} \tag{132}$$

provided $\varepsilon = \pm 1$.

(XI) If $C > 0$ and $D = 0$, then one gets the solutions:

$$u(x,t) = \left[\frac{4Cn^2 e^{\varepsilon n\sqrt{C}(x-ct)}}{-1 + 4CEn^4 e^{2\varepsilon n\sqrt{C}(x-ct)}}\right]^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta_0)} \tag{133}$$

$$v(x,t) = \chi\left[\frac{4Cn^2 e^{\varepsilon n\sqrt{C}(x-ct)}}{-1 + 4CEn^4 e^{2\varepsilon n\sqrt{C}(x-ct)}}\right]^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta_0)} \tag{134}$$

provided $\varepsilon = \pm 1$.

(XII) If $C = 0$ and $D = 0$, then one gets the solutions:

$$u(x,t) = \left[\frac{\varepsilon}{n\sqrt{E}(x-ct)}\right]^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta_0)} \tag{135}$$

$$v(x,t) = \chi\left[\frac{\varepsilon}{n\sqrt{E}(x-ct)}\right]^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta_0)} \tag{136}$$

provided $E > 0$ and $\varepsilon = \pm 1$.

Case–7: If we substitute $A = 0, B = \frac{8C^2}{27D}, E = \frac{D^2}{4C}$, in the above conditions (43)–(51), then one gets the results:

$$\begin{aligned}
&H_1 = -\left[\frac{4}{27}(5n^2+4)+1\right]C^2 - H_0 C,\\
&\Delta_1 = -\frac{13}{18}(2+n)D\left[C\left(n^2+\frac{58}{39}n+\frac{58}{39}\right)+\frac{9}{13}H_0\right],\\
&\Delta_2 = -(1+n)\frac{D^2}{4C}\left[C\left(7n^2+12n+6\right)+H_0\right],\\
&\Delta_3 = -(3n+2)(2n+1)(1+n)\frac{D^3}{4C}, \quad \Delta_4 = -(3n+1)(2n+1)(1+n)\frac{D^4}{16C^2},\\
&\Gamma_1 = \frac{4(n-2)C^2}{27D}\left[\left(n^2-2n+2\right)C+H_0\right], \quad \Gamma_2 = \frac{16(n-1)(3n^2-8n+4)C^4}{729D^2}, \quad \Gamma_3 = 0, \quad \Gamma_4 = 0.
\end{aligned} \tag{137}$$

Under the constraint conditions (137), we have the solutions:

(I) If $C < 0$, then one gets the dark soliton solutions:

$$u(x,t) = \left[-\frac{8C\tanh^2\left(\frac{n}{2}\sqrt{-\frac{C}{3}}(x-ct)\right)}{3D\left[3+\tanh^2\left(\frac{n}{2}\sqrt{-\frac{C}{3}}(x-ct)\right)\right]}\right]^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta_0)} \tag{138}$$

$$v(x,t) = \chi\left[-\frac{8C\tanh^2\left(\frac{n}{2}\sqrt{-\frac{C}{3}}(x-ct)\right)}{3D\left[3+\tanh^2\left(\frac{n}{2}\sqrt{-\frac{C}{3}}(x-ct)\right)\right]}\right]^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta_0)} \tag{139}$$

and the singular soliton solutions:

$$u(x,t)=\left[-\frac{8C\coth^2\left(\frac{n}{2}\sqrt{-\frac{C}{3}}(x-ct)\right)}{3D\left[3+\coth^2\left(\frac{n}{2}\sqrt{-\frac{C}{3}}(x-ct)\right)\right]}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{140}$$

$$v(x,t)=\chi\left[-\frac{8C\coth^2\left(\frac{n}{2}\sqrt{-\frac{C}{3}}(x-ct)\right)}{3D\left[3+\coth^2\left(\frac{n}{2}\sqrt{-\frac{C}{3}}(x-ct)\right)\right]}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{141}$$

(II) If $C>0$, then one gets the periodic solutions:

$$u(x,t)=\left[\frac{8C\tan^2\left(\frac{n}{2}\sqrt{\frac{C}{3}}(x-ct)\right)}{3D\left[3-\tan^2\left(\frac{n}{2}\sqrt{\frac{C}{3}}(x-ct)\right)\right]}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{142}$$

$$v(x,t)=\chi\left[\frac{8C\tan^2\left(\frac{n}{2}\sqrt{\frac{C}{3}}(x-ct)\right)}{3D\left[3-\tan^2\left(\frac{n}{2}\sqrt{\frac{C}{3}}(x-ct)\right)\right]}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{143}$$

or

$$u(x,t)=\left[\frac{8C\cot^2\left(\frac{n}{2}\sqrt{\frac{C}{3}}(x-ct)\right)}{3D\left[3-\cot^2\left(\frac{n}{2}\sqrt{\frac{C}{3}}(x-ct)\right)\right]}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{144}$$

$$v(x,t)=\chi\left[\frac{8C\cot^2\left(\frac{n}{2}\sqrt{\frac{C}{3}}(x-ct)\right)}{3D\left[3-\cot^2\left(\frac{n}{2}\sqrt{\frac{C}{3}}(x-ct)\right)\right]}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{145}$$

Case–8: If we substitute $A=0$ in the above conditions (43)–(51), then one gets the results

$$\begin{aligned}
&H_1=-\frac{1}{2}\left(5n^2+4\right)DB-C(C+H_0),\\
&\Delta_1=-(2+n)\left\{\left[\left(3BE+\frac{1}{2}DC\right)n^2+(n+1)(BE+DC)\right]+\frac{1}{2}DH_0\right\},\\
&\Delta_2=-(1+n)\left\{\left[n^2\left(\frac{3}{4}D^2+4EC\right)+(2n+1)\left(D^2+2EC\right)\right]+H_0E\right\},\\
&\Delta_3=-(3n+2)(2n+1)(1+n)ED,\quad \Delta_4=-(3n+1)(2n+1)(1+n)E^2,\\
&\Gamma_1=\frac{1}{2}(n-2)B\left[\left(n^2-2n+2\right)C+H_0\right],\\
&\Gamma_2=\frac{1}{4}(n-1)\left(3n^2-8n+4\right)B^2,\quad \Gamma_3=0,\quad \Gamma_4=0,
\end{aligned} \tag{146}$$

Under the constraint conditions (146), we have the solutions:

(I) If $B=\frac{D^3(m^2-1)}{32m^2E^2}$ and $C=\frac{D^2(5m^2-1)}{16m^2E}$ where $0<m<1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\varepsilon\,\mathrm{sn}\left(\frac{nD}{4m}\sqrt{\frac{1}{E}}(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{147}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\varepsilon\,\mathrm{sn}\left(\frac{nD}{4m}\sqrt{\frac{1}{E}}(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{148}$$

or

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon}{m\,\mathrm{sn}\left(\frac{nD}{4m}\sqrt{\frac{1}{E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{149}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon}{m\,\mathrm{sn}\left(\frac{nD}{4m}\sqrt{\frac{1}{E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{150}$$

provided $\varepsilon=\pm1$ and $E>0$.

(II) If $B=\frac{D^3(1-m^2)}{32E^2}$ and $C=\frac{D^2(5-m^2)}{16E}$ where $0<m<1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\varepsilon m\,\mathrm{sn}\left(\frac{nD}{4}\sqrt{\frac{1}{E}}(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{151}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\varepsilon m\,\mathrm{sn}\left(\frac{nD}{4}\sqrt{\frac{1}{E}}(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{152}$$

or

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon}{\mathrm{sn}\left(\frac{nD}{4}\sqrt{\frac{1}{E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{153}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon}{\mathrm{sn}\left(\frac{nD}{4}\sqrt{\frac{1}{E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{154}$$

provided $\varepsilon=\pm1$ and $E>0$.

(III) If $B=\frac{D^3}{32m^2E^2}$ and $C=\frac{D^2(4m^2+1)}{16m^2E}$ where $0<m<1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\varepsilon\,\mathrm{cn}\left(\frac{nD}{4m}\sqrt{-\frac{1}{E}}(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{155}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\varepsilon\,\mathrm{cn}\left(\frac{nD}{4m}\sqrt{-\frac{1}{E}}(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{156}$$

or

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon\sqrt{1-m^2}\,\mathrm{sn}\left(\frac{nD}{4m}\sqrt{-\frac{1}{E}}(x-ct)\right)}{\mathrm{dn}\left(\frac{nD}{4m}\sqrt{-\frac{1}{E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{157}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon\sqrt{1-m^2}\,\mathrm{sn}\left(\frac{nD}{4m}\sqrt{-\frac{1}{E}}(x-ct)\right)}{\mathrm{dn}\left(\frac{nD}{4m}\sqrt{-\frac{1}{E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{158}$$

provided $\varepsilon=\pm1$ and $E<0$.

(IV) If $B=\frac{m^2D^3}{32E^2(m^2-1)}$ and $C=\frac{D^2(5m^2-4)}{16E(m^2-1)}$ where $0<m<1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon}{\sqrt{1-m^2}}\mathrm{dn}\left(\frac{nD}{4}\sqrt{-\frac{1}{(1-m^2)E}}(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{159}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon}{\sqrt{1-m^2}}\mathrm{dn}\left(\frac{nD}{4}\sqrt{-\frac{1}{(1-m^2)E}}(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{160}$$

or

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon}{\mathrm{dn}\left(\frac{nD}{4}\sqrt{-\frac{1}{(1-m^2)E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{161}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon}{\mathrm{dn}\left(\frac{nD}{4}\sqrt{-\frac{1}{(1-m^2)E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{162}$$

provided $\varepsilon=\pm1$ and $E<0$.

(VI) If $B=\frac{D^3}{32E^2(1-m^2)}$ and $C=\frac{D^2(4m^2-5)}{16E(m^2-1)}$ where $0<m<1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon}{\mathrm{cn}\left(\frac{nD}{4}\sqrt{\frac{1}{(1-m^2)E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{163}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon}{\mathrm{cn}\left(\frac{nD}{4}\sqrt{\frac{1}{(1-m^2)E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{164}$$

or

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon\,\mathrm{dn}\left(\frac{nD}{4}\sqrt{\frac{1}{(1-m^2)E}}(x-ct)\right)}{\sqrt{1-m^2}\,\mathrm{sn}\left(\frac{nD}{4}\sqrt{\frac{1}{(1-m^2)E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{165}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon\,\mathrm{dn}\left(\frac{nD}{4}\sqrt{\frac{1}{(1-m^2)E}}(x-ct)\right)}{\sqrt{1-m^2}\,\mathrm{sn}\left(\frac{nD}{4}\sqrt{\frac{1}{(1-m^2)E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{166}$$

provided $\varepsilon=\pm1$ and $E>0$.

(VII) If $B=\frac{m^2D^3}{32E^2}$ and $C=\frac{D^2(m^2+4)}{16E}$ where $0<m<1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\varepsilon\,\mathrm{dn}\left(\frac{nD}{4}\sqrt{-\frac{1}{E}}(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{167}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\varepsilon\,\mathrm{dn}\left(\frac{nD}{4}\sqrt{-\frac{1}{E}}(x-ct)\right)\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{168}$$

or

$$u(x,t)=\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon\sqrt{1-m^2}}{\mathrm{dn}\left(\frac{nD}{4}\sqrt{-\frac{1}{E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{169}$$

$$v(x,t)=\chi\left[-\frac{D}{4E}\left\{1+\frac{\varepsilon\sqrt{1-m^2}}{\mathrm{dn}\left(\frac{nD}{4}\sqrt{-\frac{1}{E}}(x-ct)\right)}\right\}\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{170}$$

provided $\varepsilon=\pm1$ and $E<0$.

Case–9: If we substitute $A=E=0$ in the above conditions (43)–(51), then one gets the results:

$$\begin{gathered}
H_1=-\left[\left(\frac{5}{2}n^2+2\right)DB+C^2+H_0C\right],\\
\Delta_1=-(2+n)D\left\{\left(n^2+2n+2\right)C+H_0\right\},\quad \Delta_2=-(1+n)\left(\frac{3}{4}n^2+2n+1\right)D^2,\\
\Delta_3=0,\quad \Delta_4=0,\quad \Gamma_1=\frac{(n-2)}{2}B\left[\left(n^2-2n+2\right)C+H_0\right],\\
\Gamma_2=(n-1)\left(\frac{3}{4}n^2-2n+1\right)B^2,\quad \Gamma_3=0,\quad \Gamma_4=0.
\end{gathered} \tag{171}$$

Under the constraint conditions (171), we have the solutions:

(I) If $B=\frac{C^2m^2(m^2-1)}{D(2m^2-1)^2}$ where $0<m<1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t)=\left[-\frac{m^2C}{D(2m^2-1)}\mathrm{cn}^2\left(\frac{n}{2}\sqrt{\frac{C}{2m^2-1}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{172}$$

$$v(x,t)=\left[-\frac{m^2C}{D(2m^2-1)}\mathrm{cn}^2\left(\frac{n}{2}\sqrt{\frac{C}{2m^2-1}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{173}$$

provided $D<0$ and $C(2m^2-1)>0$.

(II) If $B=\frac{C^2m^2}{D(m^2+1)^2}$ where $0<m<1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t)=\left[-\frac{m^2C}{D(m^2+1)}\mathrm{sn}^2\left(\frac{n}{2}\sqrt{-\frac{C}{m^2+1}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{174}$$

$$v(x,t)=\chi\left[-\frac{m^2C}{D(m^2+1)}\mathrm{sn}^2\left(\frac{n}{2}\sqrt{-\frac{C}{m^2+1}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{175}$$

or

$$u(x,t)=\left[-\frac{m^2C}{D(m^2+1)}\mathrm{cd}^2\left(\frac{n}{2}\sqrt{-\frac{C}{m^2+1}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{176}$$

$$v(x,t)=\chi\left[-\frac{m^2C}{D(m^2+1)}\mathrm{cd}^2\left(\frac{n}{2}\sqrt{-\frac{C}{m^2+1}}(x-ct)\right)\right]^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{177}$$

provided $D > 0$ and $C < 0$.

(III) If $B = \frac{C^2(1-m^2)}{D(2-m^2)^2}$ where $0 < m < 1$, then one gets the Jacobi elliptic function solutions:

$$u(x,t) = \left[-\frac{C}{D(2-m^2)} \mathrm{dn}^2 \left(\frac{n}{2} \sqrt{\frac{C}{2-m^2}} (x-ct) \right) \right]^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta_0)} \tag{178}$$

$$v(x,t) = \chi \left[-\frac{C}{D(2-m^2)} \mathrm{dn}^2 \left(\frac{n}{2} \sqrt{\frac{C}{2-m^2}} (x-ct) \right) \right]^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta_0)} \tag{179}$$

provided $D < 0$ and $C > 0$. Finally, if $m \to 1^-$ or $m \to 0^+$ in the above Jacobi elliptic function solutions, we can get the hyperbolic and periodic solutions, respectively.

4. Conservation laws

In the system (2)-(3) above, we let $u = U + iW$ and $v = V + iZ$ so that the system splits into a system of four pdes whose conserved flows (T^t, T^x) are constructed using the multiplier approach. It turns out that if $Q_1 = Q_2$, we have a single multiplier $Q = (-u, -v, w, z)$ giving rise to the conserved density

$$T_1^t = \frac{1}{2}\left(U^2 + W^2 + V^2 + Z^2\right) \tag{180}$$

so that a corresponding conserved density of the complex system (2)-(3) is

$$\Phi_1^t = |u|^2 + |v|^2.$$

Unless, $\tau_i = 0, \zeta_i = 0, \eta_i = 0, \xi_i = 0, \alpha_i = 0, \beta_i = 0, \gamma_i = 0, \delta_i = 0$ and $\nu_i = -\lambda_i$, for $i = 1, 2$, No other conservation laws exist. Under these conditions, we have momentum and energy conserved.

The following conserved density, T^t, for linear momentum is

$$T_2^t = -\left[-\frac{1}{2}V_x Z - \frac{1}{2}U_x W + \frac{1}{2}Z_x U + \frac{1}{2}W_x V \right] \tag{181}$$

and the momentum density is

$$\Phi_2^t = \mathscr{I}(u^* u_x) + \mathscr{I}(v^* v_x). \tag{182}$$

Some of the terms in the conserved density corresponding to Hamiltonian are

$$\begin{aligned} T_3^t = \; & 169\left(V^2+Z^2\right)^n V^2 f_2 n^4 + 36\left(V^2+Z^2\right)^n Z^2 f_2 n^7 - 16\left(V^2+Z^2\right)^n Z^2 f_2 n - 36\left(V^2+Z^2\right)^n V^2 f_2 n^6 \\ & -36\left(V^2+Z^2\right)^n Z^2 f_2 n^6 - 169\left(V^2+Z^2\right)^n V^2 f_2 n^5 - 169\left(V^2+Z^2\right)^n Z^2 f_2 n^5 - 98\left(V^2+Z^2\right)^{2n} V^2 h_2 n^5 \\ & -9\left(V^2+Z^2\right)^{2n} Z^2 h_2 n^6 - 9\left(V^2+Z^2\right)^{2n} V^2 h_2 n^6 + 18\left(V^2+Z^2\right)^{2n} Z^2 h_2 n^7 + 18\left(V^2+Z^2\right)^{2n} V^2 h_2 n^7 \\ & -32\left(V^2+Z^2\right)^{2n} Z^2 h_2 n - 32\left(V^2+Z^2\right)^{2n} V^2 h_2 n - 56\left(V^2+Z^2\right)^{2n} Z^2 h_2 n^2 + 49\left(V^2+Z^2\right)^{2n} Z^2 h_2 n^4 \\ & \ldots \end{aligned} \tag{183}$$

It is clear that the density is not only cumbersome but when written in terms of u and v would be divergent.

The bright solitons along the two components of the magneto–optic waveguides take the form:

$$u(x,t) = \frac{A_1}{\{D + \cosh[B(x-ct)]\}^{\frac{1}{n}}} e^{i(-\kappa x+\omega t+\theta_0)} \tag{184}$$

and

$$v(x,t) = \frac{A_2}{\{D + \cosh[B(x-ct)]\}^{\frac{1}{n}}} e^{i(-\kappa x+\omega t+\theta_0)}. \tag{185}$$

Thus, for these bright solitons, the two conserved quantities would be

$$P = \int_{-\infty}^{\infty} \left(|u|^2 + |v|^2\right) dx = \frac{2}{2^{\frac{2}{n}} B} \left(A_1^2 + A_2^2\right) F\left(\frac{2}{n}, \frac{2}{n}; \frac{2}{n} + \frac{1}{2}; \frac{1-D}{2} \right) \frac{\Gamma\left(\frac{2}{n}\right)\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{n} + \frac{1}{2}\right)} \tag{186}$$

and

$$M = i\int_{-\infty}^{\infty}\left\{\left(uu_x^* - u^*u_x\right) + \left(vv_x^* - v^*v_x\right)\right\}dx = \frac{2\kappa}{2^{\frac{2}{n}}B}\left(A_1^2 + A_2^2\right)F\left(\frac{2}{n}, \frac{2}{n}; \frac{2}{n} + \frac{1}{2}; \frac{1-D}{2}\right)\frac{\Gamma\left(\frac{2}{n}\right)\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{n} + \frac{1}{2}\right)} \tag{187}$$

which represent total power and linear momentum of the solitons, respectively. Here, Gauss' hypergeometric function is written as:

$$F(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!} \tag{188}$$

where the Pochhammer symbol is:

$$(p)_n = \begin{cases} 1 & \mathrm{n} = 0, \\ p(p+1)\cdots(p+n-1) & \mathrm{n} \ > 0. \end{cases} \tag{189}$$

The condition that guarantees convergence of the series is

$$|z| < 1 \tag{190}$$

which for (186) and (187) means

$$-1 < D < 3. \tag{191}$$

Finally, Rabbe's test of convergence reveals another criterion for convergence of the series, namely

$$\gamma < \alpha + \beta \tag{192}$$

which transforms to

$$0 < n < 4. \tag{193}$$

This gives the zone of existence of the solitons which simply means the generalized KE will yield solitons in magneto–optic waveguides provided (193) stays valid. Benjamin–Fier stability analysis would also yield the same condition for the existence of solitons.

5. Conclusions

This paper revealed soliton solutions in terms of generalized KE that are recovered through an intermediate layer of solutions, namely Jacobi's elliptic functions. The limiting process eventually revealed the soliton solutions that are classified as bright, dark, singular types. Another class of solutions were also recovered, and those are in terms of Weierstrass' elliptic functions. Finally, the conservation laws are formulated. The power and linear momentum are the only two conservation laws that fall out of the model. A natural consequence of conservation laws is the stability zone for the solitons that naturally emerged from the convergence criteria of Gauss' hypergeometric functions that yielded bounds for the power law parameter n. This therefore avoided the computation of the stability zone for the solitons by the usage of Benjamin–Fier stability analysis. Pretty slick!

Later, further research activities will be carried out to secure furthermore novel and interesting results to the model utilizing a variety of integration schemes that have been successfully performed in this field of research [26–45]. Studies in this context are in progress and will be reported with time.

Declaration of Competing Interest

The authors also declare that there is no conflict of interest.

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