Cubic–quartic polarized optical solitons and conservation laws for perturbed Fokas–Lenells model

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This paper studies polarized cubic–quartic solitons that are modeled by Fokas–Lenells equation in birefringent fibers. Two integration schemes recovered a spectrum of soliton solutions to the model. Subsequently, the bright solitons compute the corresponding conserved quantities from the respective densities that are recovered by the multiplier approach.

Keywords: Solitons; polarization; Fokas–Lenells.

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1. Introduction

The concept of cubic–quartic (CQ) optical solitons first appeared a couple of years ago as a generalization to pure-quartic solitons and pure-cubic solitons that were introduced to sustain the delicate balance between chromatic dispersion (CD) and nonlinearity. The issue with pure-quartic solitons is that an analytical soliton solution cannot be established unless one is looking for a stationary soliton solution. On the other hand, pure-cubic soliton model is not integrable unless Hamiltonian perturbation terms are included in it. This led to the novel concept of CQ solitons that has achieved success in the fields of quantum optics and telecommunication industry. In CQ solitons, CD is replaced by a combination of third-order dispersion (3OD) and fourth-order dispersion (4OD) that can together replenish the low count of CD. A plethora of results are reported during the past decade or so in these arenas.1–27

One of the familiar models that addresses the dynamics of soliton propagation through optical fibers and PCF, apart from the familiar nonlinear Schrödinger’s equation, is Fokas–Lenells equation (FLE). This model is well studied in the context of polarization-preserving fibers with CD and/or CQ replacing CD. Today’s work turns the page to extend the study of CQ solitons with FLE for polarized pulses. Thus, the model equations for CQ solitons in birefringent fibers are first introduced. Subsequently, a couple of familiar integration algorithms would give way to a spectrum of CQ soliton solutions to FLE in birefringent fibers. Finally, the paper enumerates and exhibits conservation laws to the model that are recovered by the aid of multiplier approach. The corresponding conserved quantities are eventually computed and presented. These results are displayed after a succinct intro to the governing model.

1.1. Governing model

The CQ in birefringent fibers for the perturbed FLE reads as follows:

\[ iq_t + ia_1 q_{xxx} + b_1 q_{xxxx} + (c_1 |q|^2 + d_1 |r|^2)(e_1 q + i f_1 q_x) + qr^* (\gamma_1 r + i \eta_1 r_x) = i[\delta_1 q_x + \lambda_1 (|q|^2 q)_x + \mu_1 (|q|^2)_{xx} q], \tag{1} \]

\[ ir_t + ia_2 r_{xxx} + b_2 r_{xxxx} + (c_2 |r|^2 + d_2 |q|^2)(e_2 r + i f_2 r_x) + qr^* (\gamma_2 q + i \eta_2 q_x) = i[\delta_2 r_x + \lambda_2 (|r|^2 r)_x + \mu_2 (|r|^2)_{xx} r], \tag{2} \]

where \( a_j, \ b_j, \ c_j, \ d_j, \ e_j, \ f_j, \ \gamma_j, \ \eta_j, \ \delta_j, \ \lambda_j \) and \( \mu_j \) for \( j = 1, 2 \) are all real-valued constants and \( i = \sqrt{-1} \), while the independent variables \( x \) and \( t \) represent spatial and temporal variables, respectively. The dependent variables \( q = q(x, t) \) and \( r = r(x, t) \) are the complex valued describing the wave profiles for the two components in polarization-preserving fibers. The first terms in Eqs. (1) and (2) are linear temporal evolution of the pulses. Next, the coefficients of \( a_j \) are the 3ODs and the coefficients \( b_j \)
stand for the 4ODs. Then, $c_j$ are the self-phase modulation (SPM) terms while $d_j$ stand for cross-phase modulation (XPM). Next, $f_j$ represent nonlinear dispersions, while $q_j$ are four wave-mixing (4WM) terms. The parameters $e_j$ coupled with $c_j$ give SPM and when coupled with $d_j$ give XPM. Then $q_j$ give the XPM along with 4WM. Finally, $\delta_j$ are the coefficients of inter-modal dispersion (IMD) and the parameters $\lambda_j$ give self-steepening terms (SSTs) that avoid the formation of shock waves, while $\mu_j$ are nonlinear dispersion coefficients.

This paper is organized as follows. In Sec. 2, the mathematical preliminaries are introduced. In Secs. 3 and 4, the generalized auxiliary equation method and the addendum to Kudryashov’s method are successfully applied. In Sec. 6, the conservation laws of the model are addressed with a few conclusive words that are given in Sec. 7.

2. Mathematical Preliminaries

In order to solve Eqs. (1) and (2), we assume the hypothesis:

$$q(x,t) = U_1(\xi) \exp[i\psi(x,t)],$$
$$r(x,t) = U_2(\xi) \exp[i\psi(x,t)],$$

such that

$$\xi = x - vt, \quad \psi(x,t) = -\kappa x + \omega t + \theta_0,$$

where $v, \kappa, \omega$ and $\theta_0$ are all nonzero constants to be determined which represent soliton velocity, soliton frequency, wave number and phase constant, respectively. Next, $\psi(x,t)$ is a real function which represents the phase component of the soliton, while $U_j(\xi)$ are real functions which represent the shape of the pulse of the solitons. Substituting (3) along with (4) into Eqs. (1) and (2), separating the real and imaginary parts, we deduce that the real parts are

$$b_1 U_1''' + (3a_1\kappa - 6b_1\kappa^2)U_1'' - [\omega + \delta_1\kappa + a_1\kappa^3 - b_1\kappa^4]U_1$$
$$+ (e_1 + f_1\kappa)(c_1 U_1^2 + d_1 U_1^2)U_1 + (\gamma_1 + \eta_1\kappa)U_1 U_2^2 - \lambda_1\kappa U_1^3 = 0$$

and

$$b_2 U_2''' + (3a_2\kappa - 6b_2\kappa^2)U_2'' - [\omega + \delta_2\kappa + a_2\kappa^3 - b_2\kappa^4]U_2$$
$$+ (e_2 + f_2\kappa)(c_2 U_2^2 + d_2 U_1^2)U_2 + (\gamma_2 + \eta_2\kappa)U_2 U_1^2 - \lambda_2\kappa U_2^3 = 0,$$

while the imaginary parts are

$$(a_1 - 4b_1\kappa)U_1''' - (\delta_1 + v + 3a_1\kappa^2 - 4b_1\kappa^3)U_1' - (3\lambda_1 + 2\mu_1)U_1^2 U_1'$$
$$+ f_1(c_1 U_1^2 + d_1 U_1^2)U_1' + \eta_1 U_1 U_2 U_2' = 0$$

(7)
and
\[
(a_2 - 4b_2 \kappa)U''_2 - (\delta_2 + v + 3a_2 \kappa^2 - 4b_2 \kappa^3)U'_2 - (3\lambda_2 + 2\mu_2)U^2_2 U'_2 \\
+ f_2(c_2 U^2_2 + d_2 U'_2)U'_2 + \eta_2 U_2 U'_1 U'_1 = 0.
\]
(8)

Set
\[
U_2(\xi) = \chi U_1(\xi),
\]
(9)
where \(\chi\) is a nonzero constant, such that \(\chi \neq 1\). Consequently, the real parts change to
\[
b_1 U''_1 + (3a_1 \kappa - 6b_1 \kappa^2)U''_1 - [\omega + \delta_1 \kappa + a_1 \kappa^3 - b_1 \kappa^4]U_1 \\
+ [(e_1 + f_1 \kappa)(c_1 + d_1 \chi^2) + (\gamma_1 + \eta_1 \kappa)\chi^2 - \lambda_1 \kappa]U^2_1 = 0
\]
(10)
and
\[
\chi b_2 U''_1 + (3a_2 \kappa - 6b_2 \kappa^2)\chi U''_1 - [\omega + \delta_2 \kappa + a_2 \kappa^3 - b_2 \kappa^4]\chi U_1 \\
+ [(e_2 + f_2 \kappa)(c_2 \chi^2 + d_2)\chi + (\gamma_2 + \eta_2 \kappa)\chi - \lambda_2 \kappa \chi^3]U^2_1 = 0,
\]
(11)
while the imaginary parts become
\[
(a_1 - 4b_1 \kappa)U''_1 - (\delta_1 + v + 3a_1 \kappa^2 - 4b_1 \kappa^3)U'_1 \\
+ [f_1(c_1 + d_1 \chi^2) - (3\lambda_1 + 2\mu_1) + \eta_1 \chi^2]U^2_1 U'_1 = 0
\]
(12)
and
\[
(a_2 - 4b_2 \kappa)\chi U''_1 - (\delta_2 + v + 3a_2 \kappa^2 - 4b_2 \kappa^3)\chi U'_1 \\
+ [f_2(c_2 \chi^2 + d_2)\chi - (3\lambda_2 + 2\mu_2)\chi^2 + \eta_2 \chi]U^2_1 U'_1 = 0.
\]
(13)
The linearly independent principle is applied on (12) and (13) to get the frequency of soliton:
\[
\kappa = \frac{a_j}{4b_j}, \quad a_j \neq 0, \quad b_j \neq 0 \quad \text{for} \quad j = 1, 2,
\]
(14)
the velocity of soliton is given by
\[
v = 4b_j \kappa^3 - \delta_j - 3a_j \kappa^2 \quad \text{for} \quad j = 1, 2,
\]
(15)
and the constraint conditions
\[
f_1(c_1 + d_1 \chi^2) - (3\lambda_1 + 2\mu_1) + \eta_1 \chi^2 = 0,
\]
(16)
\[
f_2(c_2 \chi^2 + d_2)\chi - (3\lambda_2 + 2\mu_2)\chi^2 + \eta_2 = 0.
\]
(16)
From (14) and (15), one gets the natural constraint relations as
\[
a_1 b_2 = a_2 b_1,
\]
\[
4b_1 \kappa^3 - \delta_1 - 3a_1 \kappa^2 = 4b_2 \kappa^3 - \delta_2 - 3a_2 \kappa^2.
\]
(17)
Equations (10) and (11) have the same form under the constraint conditions:

\[ b_1 = \chi b_2, \]

\[ 3a_1 \kappa - 6b_1 \kappa^2 = (3a_2 \kappa - 6b_2 \kappa^2) \chi, \]

\[ \omega + \delta_1 \kappa + a_1 \kappa^3 - b_1 \kappa^4 = (\omega + \delta_2 \kappa + a_2 \kappa^3 - b_2 \kappa^4) \chi, \]

\[ (e_1 + f_1 \kappa)(c_1 + d_1 \chi^2) + (\gamma_1 + \eta_1 \kappa) \chi^2 - \lambda_1 \kappa \]

\[ = (e_2 + f_2 \kappa)(c_2 \chi^2 + d_2) \chi + (\gamma_2 + \eta_2 \kappa) \chi - \lambda_2 \kappa \chi^3. \]

From (18), we have the wave number of the soliton as

\[ \omega = \frac{\delta_1 \kappa + a_1 \kappa^3 - b_1 \kappa^4 - (\delta_2 \kappa + a_2 \kappa^3 - b_2 \kappa^4) \chi}{(\chi - 1)}. \]  

Equation (10) can be rewritten as

\[ b_1 U_1'''' + \Pi_0 U_1'' + \Pi_1 U_1 + \Pi_3 U_1^3 = 0, \]  

where

\[
\begin{align*}
\Pi_0 &= 3a_1 \kappa - 6b_1 \kappa^2, \\
\Pi_1 &= -[\omega + \delta_1 \kappa + a_1 \kappa^3 - b_1 \kappa^4], \\
\Pi_3 &= (e_1 + f_1 \kappa)(c_1 + d_1 \chi^2) + (\gamma_1 + \eta_1 \kappa) \chi^2 - \lambda_1 \kappa.
\end{align*}
\]

Now, we will solve Eq. (20) using the following methods.

3. The Generalized Auxiliary Equation Method

According to this method, we assume that Eq. (20) has the formal solution:

\[ U_1(\xi) = \sum_{l=0}^{N} \alpha_l f^l(\xi), \]

where \( f(\xi) \) satisfies the first-order auxiliary equation:

\[ f^2(\xi) = \sum_{m=0}^{6} h_m f^m(\xi). \]

Here, \( \alpha_l (l = 0, 1, \ldots, N) \) and \( h_m (m = 0, 1, \ldots, 6) \) are constants to be determined such that \( \alpha_N \neq 0 \) and \( h_0 \neq 0 \), where \( N \) is a positive integer. Balancing \( U_1'''' \) and \( U_1^3 \) in Eq. (20), one gets the balance number \( N = 4 \). From (22), one gets the formal solution of Eq. (20) as

\[ U_1(\xi) = \alpha_0 + \alpha_1 f(\xi) + \alpha_2 f^2(\xi) + \alpha_3 f^3(\xi) + \alpha_4 f^4(\xi), \]

where \( \alpha_l (l = 0, 1, 2, 3, 4) \) are constants to be determined such that \( \alpha_4 \neq 0 \). Substituting (24) and (23) with \( h_0 = h_1 = h_3 = h_5 = 0 \) into Eq. (20), collecting all the coefficients of \([ f(\xi) ]^m, [ f(\xi) ]^2, [ f(\xi) ]^3, [ f(\xi) ]^4 \) \((m_l = 0, 1, \ldots, 12, m_2 = 0, 1)\) and setting these coefficients to zero, we have a set of algebraic equations which can be solved using the
Maple to obtain the following results:

\[
\begin{align*}
\Pi_3 \alpha_4^3 + 1920b_1 \alpha_4 h_6^2 &= 0, \\
3\Pi_3 \alpha_3 \alpha_4^2 + 945b_1 \alpha_3 h_6^2 &= 0, \\
3\Pi_3 \alpha_2 \alpha_4^2 + 384b_1 \alpha_2 h_6^2 + 3\Pi_3 \alpha_3^2 \alpha_4 + 2688b_1 \alpha_4 h_6 h_1 &= 0, \\
105b_1 \alpha_1 h_6^2 + 1260b_1 \alpha_4 \alpha_3 h_6 + 6\Pi_3 \alpha_3 \alpha_3 \alpha_4 + 3\Pi_3 \alpha_1 \alpha_3^2 + \Pi_3 \alpha_3^3 &= 0, \\
6\Pi_3 \alpha_1 \alpha_3 \alpha_4 + 3\Pi_3 \alpha_2 \alpha_3^2 + 24\Pi_0 \alpha_4 h_6 + 3\Pi_3 \alpha_3 \alpha_2^2 + 1920b_1 \alpha_4 h_6 h_2 & + 840b_1 \alpha_4 h_6^2 + 3\Pi_3 \alpha_3^2 \alpha_4 + 480b_1 \alpha_4 h_2 h_6 &= 0, \\
870b_1 \alpha_3 h_6 \alpha_3 h_2 + 6\Pi_3 \alpha_1 \alpha_2 \alpha_4 + 15\Pi_0 \alpha_3 h_6 + 3\Pi_3 \alpha_1 \alpha_3^2 + 360b_1 \alpha_3 h_4^2 & + 6\Pi_3 \alpha_0 \alpha_3 \alpha_4 + 120b_1 \alpha_4 h_1 h_6 + 3\Pi_3 \alpha_3 \alpha_3 \alpha_3 &= 0, \\
320b_1 \alpha_2 h_6 \alpha_2 h_2 + 6\Pi_3 \alpha_0 \alpha_2 \alpha_4 + 6\Pi_3 \alpha_1 \alpha_2 \alpha_3 + 3\Pi_3 \alpha_3 \alpha_2 \alpha_4 + 20\Pi_0 \alpha_4 h_4 & + 8\Pi_0 \alpha_2 h_6 + \Pi_3 \alpha_3^2 + 2 + 120b_1 \alpha_2 h_4^2 + 3\Pi_3 \alpha_0 \alpha_3^2 + 1040b_1 \alpha_4 h_4 h_2 &= 0, \\
24b_1 \alpha_1 h_4^2 + 78b_1 \alpha_1 h_1 h_2 + 6\Pi_3 \alpha_0 \alpha_2 \alpha_3 + 6\Pi_3 \alpha_0 \alpha_1 \alpha_4 + 408b_1 \alpha_4 h_3 h_2 & + 3\Pi_3 \alpha_1 \alpha_2^2 + 12\Pi_0 \alpha_3 h_4 + 3\Pi_0 \alpha_1 h_6 + 3\Pi_3 \alpha_3 \alpha_3 &= 0, \\
120b_1 \alpha_4 \alpha_2 h_2 + 6\Pi_3 \alpha_0 \alpha_1 \alpha_3 + \Pi_1 \alpha_4 + 6\Pi_0 \alpha_2 h_2 + 256b_1 \alpha_4 h_2^2 & + 3\Pi_3 \alpha_3 \alpha_4 + 3\Pi_3 \alpha_0 \alpha_2^2 + 3\Pi_3 \alpha_1 \alpha_2 + 16\Pi_0 \alpha_4 h_2 &= 0, \\
20b_1 \alpha_4 \alpha_1 h_2 + 9\Pi_0 \alpha_3 h_2 + 6\Pi_3 \alpha_0 \alpha_1 \alpha_2 + 81b_1 \alpha_3 h_2 + 3\Pi_3 \alpha_0 \alpha_2 \alpha_3 & + \Pi_3 \alpha_3^2 + 2\Pi_0 \alpha_1 h_4 + \Pi_1 \alpha_3 &= 0, \\
3\Pi_3 \alpha_0 \alpha_2^2 + 16b_1 \alpha_2 h_2^2 + 3\Pi_3 \alpha_0 \alpha_1^2 + 4\Pi_0 \alpha_2 h_2 + \Pi_1 \alpha_2 &= 0, \\
3\Pi_3 \alpha_0 \alpha_1 + b_1 \alpha_1 h_2^2 + \Pi_0 \alpha_1 h_2 + \Pi_1 \alpha_1 &= 0, \\
\Pi_1 \alpha_0 + \Pi_3 \alpha_0^2 &= 0.
\end{align*}
\]

(25)

On solving the above algebraic equations (25) by using the Maple, one gets the following two results:

Result 1:

\[
\begin{align*}
\alpha_0 &= 0, & \alpha_1 &= 0, & \alpha_2 &= 4\epsilon \sqrt{\frac{6\Pi_0 h_6}{\Pi_3}}, & \alpha_3 &= 0, & \alpha_4 &= 8\epsilon h_6 \sqrt{-\frac{30b_1}{\Pi_3}}, \\
h_2 &= -\frac{\Pi_0}{20b_1}, & h_4 &= -\epsilon \sqrt{-\frac{\Pi_0 h_6}{5b_1}}, & h_6 &= h_6
\end{align*}
\]

(26)

and

\[
\Pi_1 = \frac{4\Pi_0^2}{25b_1},
\]

(27)

provided \(\Pi_0 \Pi_3 h_6 > 0\), \(\Pi_0 b_1 h_6 < 0\), \(b_1 \Pi_3 < 0\) and \(\epsilon = \pm 1\). It is well known that Eq. (23) has the following solutions: When \(h_0 = h_1 = h_3 = h_5 = 0\), we have

\[
f(\xi) = \left[-\frac{h_2 h_4 \text{sech}^2(\sqrt{h_2} \xi)}{h_4^2 - h_2 h_6 [1 + \epsilon \tanh(\sqrt{h_2} \xi)]^2}\right]^{\frac{1}{2}},
\]

(28)
provided $h_2 > 0$, $\epsilon = \pm 1$,

$$f(\xi) = \left[ \frac{h_2 h_4 \operatorname{csch}^2(\sqrt{h_2} \xi)}{h_4^2 - h_2 h_6 [1 + \epsilon \coth(\sqrt{h_2} \xi)]^2} \right]^{\frac{1}{2}}, \quad (29)$$

and

$$f(\xi) = \frac{h_2 \operatorname{sech}^2(\sqrt{h_2} \xi)}{h_4 + 2 \epsilon \sqrt{h_2} h_6 \tanh(\sqrt{h_2} \xi)}^{\frac{1}{2}}, \quad (30)$$

provided $h_2 > 0$, $h_6 > 0$ and $\epsilon = \pm 1$.

Substituting (26) along with (28)–(31) into (24), one gets the following soliton solutions of the system (1) and (2):

$$q(x, t) = \pm \frac{4}{5} \Pi_0 \sqrt{-\frac{30}{b_1 \Pi_3}} \frac{\operatorname{sech}^2 \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right)}{4 - \left[ 1 + \epsilon \tanh \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right) \right]^2}$$

$$\left[ 1 - \frac{2 \operatorname{sech}^2 \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right)}{4 - \left[ 1 + \epsilon \tanh \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right) \right]^2} e^{i(-\kappa x + \omega t + \theta_0)} \right], \quad (32)$$

and

$$r(x, t) = \pm \frac{4}{5} \chi \Pi_0 \sqrt{-\frac{30}{b_1 \Pi_3}} \frac{\operatorname{sech}^2 \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right)}{4 - \left[ 1 + \epsilon \tanh \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right) \right]^2}$$

$$\left[ 1 - \frac{2 \operatorname{sech}^2 \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right)}{4 - \left[ 1 + \epsilon \tanh \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right) \right]^2} e^{i(-\kappa x + \omega t + \theta_0)} \right], \quad (33)$$

or

$$q(x, t) = \pm \frac{4}{5} \Pi_0 \sqrt{-\frac{30}{b_1 \Pi_3}} \frac{\operatorname{csch}^2 \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right)}{4 - \left[ 1 + \epsilon \coth \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right) \right]^2}$$

$$\left[ 1 + \frac{2 \operatorname{csch}^2 \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right)}{4 - \left[ 1 + \epsilon \coth \left( \sqrt{-\frac{\Pi_0}{20b_7}} (x - vt) \right) \right]^2} e^{i(-\kappa x + \omega t + \theta_0)} \right], \quad (34)$$
and

\[
r(x, t) = \pm \frac{4}{5} \chi \Pi_0 \sqrt{-\frac{30}{b_1 \Pi_3}} \frac{\text{csch}^2\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}{4 - \left[1 + \epsilon \coth\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)\right]^2} \\
\left[1 + \frac{2\text{csch}^2\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}{4 - \left[1 + \epsilon \coth\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)\right]^2}\right] e^{i(-\kappa x + \omega t + \theta_0)},
\]

(35)

or

\[
q(x, t) = \pm \frac{1}{2} \Pi_0 \sqrt{-\frac{6}{5b_1 \Pi_3}} \left[\frac{\text{sech}^2\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}{1 + \epsilon \tanh\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}\right] \\
\left[2 - \frac{\text{sech}^2\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}{1 + \epsilon \tanh\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}\right] e^{i(-\kappa x + \omega t + \theta_0)},
\]

(36)

and

\[
r(x, t) = \pm \frac{1}{2} \chi \Pi_0 \sqrt{-\frac{6}{5b_1 \Pi_3}} \left[\frac{\text{sech}^2\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}{1 + \epsilon \tanh\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}\right] \\
\left[2 - \frac{\text{sech}^2\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}{1 + \epsilon \tanh\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}\right] e^{i(-\kappa x + \omega t + \theta_0)},
\]

(37)

or

\[
q(x, t) = \pm \frac{1}{2} \Pi_0 \sqrt{-\frac{6}{5b_1 \Pi_3}} \left[\frac{\text{csch}^2\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}{1 + \epsilon \coth\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}\right] \\
\left[2 + \frac{\text{csch}^2\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}{1 + \epsilon \coth\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}\right] e^{i(-\kappa x + \omega t + \theta_0)},
\]

(38)

and

\[
r(x, t) = \pm \frac{1}{2} \chi \Pi_0 \sqrt{-\frac{6}{5b_1 \Pi_3}} \left[\frac{\text{csch}^2\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}{1 + \epsilon \coth\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}\right] \\
\left[2 + \frac{\text{csch}^2\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}{1 + \epsilon \coth\left(\sqrt{-\frac{\Pi_0}{20b_1}} (x - vt)\right)}\right] e^{i(-\kappa x + \omega t + \theta_0)},
\]

(39)

provided \(b_1 \Pi_3 < 0\) and \(b_1 \Pi_0 < 0\).
The solutions (32)–(39) exist under the constraint condition (27).

Result 2:

\[ \alpha_0 = 0, \quad \alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0, \quad \alpha_4 = 8\epsilon h_6 \sqrt{\frac{30b_1}{\Pi_3}}, \]

\[ h_2 = -\frac{\Pi_0}{80b_1}, \quad h_4 = 0, \quad h_6 = h_6 \]

and

\[ \Pi_1 = \frac{4\Pi_0^2}{25b_1}, \]

provided \( b_1 \Pi_3 < 0 \) and \( \epsilon = \pm 1 \). It is well known that Eq. (23) has the following solutions:

When \( h_0 = h_1 = h_3 = h_5 = 0 \), we have

\[ f(\xi) = \left[ \frac{2h_2}{\epsilon \sqrt{h_4^2 - 4h_2h_6 \cosh (2\sqrt{h_2} \xi) - h_4}} \right]^{\frac{1}{\epsilon}}, \]

provided \( h_2 > 0, h_4^2 - 4h_2h_6 > 0 \) and \( \epsilon = \pm 1 \),

\[ f(\xi) = \left[ \frac{2h_2}{\epsilon \sqrt{-(h_4^2 - 4h_2h_6) \sinh (2\sqrt{h_2} \xi) - h_4}} \right]^{\frac{1}{\epsilon}}, \]

provided \( h_2 > 0, h_4^2 - 4h_2h_6 < 0 \) and \( \epsilon = \pm 1 \),

\[ f(\xi) = \left[ \frac{2h_2}{\epsilon \sqrt{h_4^2 - 4h_2h_6 \cos (2\sqrt{-h_2} \xi) - h_4}} \right]^{\frac{1}{\epsilon}}, \]

provided \( h_2 < 0, h_4^2 - 4h_2h_6 > 0 \) and \( \epsilon = \pm 1 \). Substituting (40) along with (42)–(45) into (24), one gets the following solutions of the system (1) and (2):

(I) The bright soliton solutions:

\[ q(x, t) = \pm \frac{1}{10} \Pi_0 \sqrt{\frac{30}{b_1 \Pi_3}} \text{sech}^2 \left( \frac{1}{2} \sqrt{-\frac{\Pi_0}{5b_1}} (x - vt) \right) e^{i(-\kappa x + \omega t + \theta_0)} \]

and

\[ r(x, t) = \pm \frac{1}{10} \chi \Pi_0 \sqrt{\frac{30}{b_1 \Pi_3}} \text{sech}^2 \left( \frac{1}{2} \sqrt{-\frac{\Pi_0}{5b_1}} (x - vt) \right) e^{i(-\kappa x + \omega t + \theta_0)}, \]

provided \( b_1 \Pi_3 < 0 \) and \( b_1 \Pi_0 < 0 \).
(II) The singular soliton solutions:

\[
q(x,t) = \pm \frac{1}{10} \Pi_0 \sqrt{-\frac{30}{b_1 \Pi_3}} \csc^2 \left( \frac{1}{2} \sqrt{\frac{\Pi_0}{5 b_1}} (x - vt) \right) e^{i(-\kappa x + \omega t + \theta_0)}
\]  

(48)

and

\[
r(x,t) = \pm \frac{1}{10} \chi \Pi_0 \sqrt{-\frac{30}{b_1 \Pi_3}} \csc^2 \left( \frac{1}{2} \sqrt{\frac{\Pi_0}{5 b_1}} (x - vt) \right) e^{i(-\kappa x + \omega t + \theta_0)},
\]  

(49)

provided \( b_1 \Pi_3 < 0 \) and \( b_1 \Pi_0 < 0 \).

(III) The periodic wave solutions:

\[
q(x,t) = \pm \frac{1}{10} \Pi_0 \sqrt{-\frac{30}{b_1 \Pi_3}} \sec^2 \left( \frac{1}{2} \sqrt{\frac{\Pi_0}{5 b_1}} (x - vt) \right) e^{i(-\kappa x + \omega t + \theta_0)}
\]  

(50)

and

\[
r(x,t) = \pm \frac{1}{10} \chi \Pi_0 \sqrt{-\frac{30}{b_1 \Pi_3}} \sec^2 \left( \frac{1}{2} \sqrt{\frac{\Pi_0}{5 b_1}} (x - vt) \right) e^{i(-\kappa x + \omega t + \theta_0)},
\]  

(51)

or

\[
q(x,t) = \pm \frac{1}{10} \Pi_0 \sqrt{-\frac{30}{b_1 \Pi_3}} \csc^2 \left( \frac{1}{2} \sqrt{\frac{\Pi_0}{5 b_1}} (x - vt) \right) e^{i(-\kappa x + \omega t + \theta_0)}
\]  

(52)

and

\[
r(x,t) = \pm \frac{1}{10} \chi \Pi_0 \sqrt{-\frac{30}{b_1 \Pi_3}} \csc^2 \left( \frac{1}{2} \sqrt{\frac{\Pi_0}{5 b_1}} (x - vt) \right) e^{i(-\kappa x + \omega t + \theta_0)},
\]  

(53)

provided \( b_1 \Pi_3 < 0 \) and \( b_1 \Pi_0 > 0 \).

The solutions (46)–(53) exist under the constraint condition (41).

4. An Addendum to Kudryashov’s Method

According to this method, we assume that Eq. (20) has the formal solution

\[
U_1(\xi) = \sum_{l=0}^{N} B_l R^l(\xi),
\]  

(54)

where \( B_l (l = 0, 1, 2, \ldots, N) \) are constants to be determined, such that \( B_N \neq 0 \), while \( R(\xi) \) satisfies the first-order auxiliary ordinary differential equation (ODE):

\[
R^\theta(\xi) = R^2(\xi)[1 - \Gamma R^{2\theta}(\xi)] \ln^2 K, \quad 0 < K \neq 1,
\]  

(55)
where $\Gamma$ is a constant. It is easy to show that Eq. (55) has the following solution:

$$
R(\xi) = \left[ \frac{4A}{4A^2 \exp_K(p\xi) + \Gamma \exp_K(-p\xi)} \right]^\frac{1}{p},
$$

(56)

where $A$ is a nonzero constant, $p$ is a positive integer and $\exp_K(p\xi) = K^{p\xi}$. We determine the positive integer $N$ in (54) by using the homogeneous balance method as follows.

If $D(U_1) = N, D(U_1') = N + p, D(U_1'') = N + 2p$, then we have $D[U_1^{(4)}] = N(r + 1) + ps$.

Now, by balancing $U_1'''$ with $U_1^2$ in Eq. (20), one gets

$$
N + 4p = 3N \Rightarrow N = 2p.
$$

(57)

Next, we will discuss the following cases.

**Case 1.** If we choose $p = 1$, then $N = 2$. Thus, from (54) we deduce that Eq. (20) has the formal solution:

$$
U_1(\xi) = B_0 + B_1 R(\xi) + B_2 R^2(\xi),
$$

(58)

where $B_0, B_1$ and $B_2$ are constants to be determined, such that $B_2 \neq 0$ while $R(\xi)$ satisfies the first-order auxiliary ODE:

$$
R'(\xi) = R^2(\xi)[1 - \Gamma R^2(\xi)] \ln^2 K, \quad 0 < K \neq 1,
$$

(59)

where $\Gamma$ is a constant. Substituting (58) along with (59) into Eq. (20), collecting all the coefficients of each power of $[R(\xi)]^m [R'(\xi)]^{m_2} (m_1 = 0, 1, \ldots, 6, m_2 = 0, 1)$ and setting each of these coefficients to zero, we have a system of algebraic equations which can be solved by using the Maple to get the results:

$$
B_0 = 0, \quad B_1 = 0, \quad B_2 = \pm 2\Gamma \sqrt{-\frac{30b_1}{\Pi_3}} \ln^2 K
$$

(60)

and

$$
\Pi_0 = -20b_1 \ln^2 K, \quad \Pi_1 = 64b_1 \ln^4 K,
$$

(61)

provided $b_1 \Pi_3 < 0$. Substituting (60) along with (56) into Eq. (58), one gets the solutions of the system (1) and (2) in the forms:

$$
q(x,t) = \pm 2\Gamma \sqrt{-\frac{30b_1}{\Pi_3}} \left[ \frac{4A \ln K}{4A^2 K(x-ct) + \Gamma K^-(x-ct)} \right]^2 e^{i(-\kappa x + \omega t + \theta_0)}
$$

(62)

and

$$
r(x,t) = \pm 2\chi \Gamma \sqrt{-\frac{30b_1}{\Pi_3}} \left[ \frac{4A \ln K}{4A^2 K(x-ct) + \Gamma K^-(x-ct)} \right]^2 e^{i(-\kappa x + \omega t + \theta_0)}.
$$

(63)
In particular, if we set $\Gamma = 4A^2$ in (62) and (63), then the system (1) and (2) has the bright soliton solutions as

$$q(x, t) = \pm 2\sqrt{-30b_1/\Pi_3} (\ln^2 K) \text{sech}^2[(x - ct) \ln K] e^{i(-\kappa x + \omega t + \theta_0)},$$

and

$$r(x, t) = \pm 2\chi \sqrt{-30b_1/\Pi_3} (\ln^2 K) \text{sech}^2[(x - ct) \ln K] e^{i(-\kappa x + \omega t + \theta_0)},$$

while, if $\Gamma = -4A^2$, one gets the singular soliton solution of the system (1) and (2) as

$$q(x, t) = \pm 2\sqrt{-30b_1/\Pi_3} (\ln^2 K) \text{csch}^2[(x - ct) \ln K] e^{i(-\kappa x + \omega t + \theta_0)},$$

and

$$r(x, t) = \pm 2\chi \sqrt{-30b_1/\Pi_3} (\ln^2 K) \text{csch}^2[(x - ct) \ln K] e^{i(-\kappa x + \omega t + \theta_0)}.$$  

The solutions (62)–(67) exist under the conditions (61).

**Case 2.** If we choose $p = 2$, then $N = 4$. Thus, we deduce from (54) that Eq. (20) has the formal solution:

$$U_1(\xi) = B_0 + B_1 R(\xi) + B_2 R^2(\xi) + B_3 R^3(\xi) + B_4 R^4(\xi),$$

where $B_0, B_1, B_2, B_3$ and $B_4$ are constants to be determined, such that $B_4 \neq 0$ and the function $R(\xi)$ satisfies the first-order auxiliary ODE

$$R^{\xi}(\xi) = R^2(\xi)[1 - \Gamma R^4(\xi)] \ln^2 K, \quad 0 < K \neq 1,$$

where $\Gamma$ is a constant. Substituting (68) along with (69) into Eq. (20), collecting all the coefficients of each power of $[R(\xi)]^{m_1} [R^\xi(\xi)]^{m_2}$ ($m_1 = 0, 1, 2, \ldots, 12$, $m_2 = 0, 1$) and setting each of these coefficients to zero, we have a system of algebraic equations which can be solved by using the Maple to get the results:

$$B_0 = 0, \quad B_1 = 0, \quad B_2 = 0, \quad B_3 = 0, \quad B_4 = \pm 8\Gamma \sqrt{-30b_1 \ln^2 K},$$

and

$$\Pi_0 = -80b_1 \ln^2 K, \quad \Pi_1 = 1024b_1 \ln^4 K,$$

provided $b_1 \Pi_3 < 0$. Substituting (70) along with (56) into Eq. (68), one gets the solutions of the system (1) and (2) in the forms:

$$q(x, t) = \pm 8\Gamma \sqrt{-30b_1/\Pi_3} \left[\frac{4A \ln K}{4A^2 K^2(x - ct) + \Gamma K^{-2}(x - ct)}\right]^2 e^{i(-\kappa x + \omega t + \theta_0)}.$$
and
\[ r(x, t) = \pm 8\chi \sqrt{-\frac{30b_1}{\Pi_3}} \left[ \frac{4A \ln K}{4A^2 K^{2(x-ct)} + 1} K^{-2(x-ct)} \right]^2 e^{i(-\kappa x + \omega t + \theta_0)}. \quad (73) \]

In particular, if we set \( \Gamma = 4A^2 \) in (72) and (73), then we have the bright soliton solution of the system (1) and (2) as
\[ q(x, t) = \pm 8 \sqrt{-\frac{30b_1}{\Pi_3}} (\ln^2 K) \sech^2[2(x - ct) \ln K] e^{i(-\kappa x + \omega t + \theta_0)} \quad (74) \]
and
\[ r(x, t) = \pm 8\chi \sqrt{-\frac{30b_1}{\Pi_3}} (\ln^2 K) \sech^2[2(x - ct) \ln K] e^{i(-\kappa x + \omega t + \theta_0)}, \quad (75) \]
while, if \( \Gamma = -4A^2 \), one gets the singular soliton solution of the system (1) and (2) as
\[ q(x, t) = \pm 8 \sqrt{-\frac{30b_1}{\Pi_3}} (\ln^2 K) \csch^2[2(x - ct) \ln K] e^{i(-\kappa x + \omega t + \theta_0)} \quad (76) \]
and
\[ r(x, t) = \pm 8\chi \sqrt{-\frac{30b_1}{\Pi_3}} (\ln^2 K) \csch^2[2(x - ct) \ln K] e^{i(-\kappa x + \omega t + \theta_0)}. \quad (77) \]
The solutions (72)–(77) exist under the conditions (71).

Similarly, we can find many other solutions by choosing other values for \( p \) and \( N \).

5. Conservation Laws

Following some detailed calculations, we get that system (1) and (2) admits three conservation laws corresponding to “power”, “linear momentum” and “Hamiltonian” for particular cases of the parameters; the conserved densities are, respectively, given as follows:

(i) If \( \eta_1 = \eta_2 \) and \( d_1 = d_2 = 0 \), we get the conserved “power density”
\[ T^I_c = \frac{1}{2} (|q|^2 + |r|^2). \quad (78) \]

(ii) If \( \eta_1 = \eta_2, d_1 = d_2 = 0, \lambda_1 = -\mu_2, \lambda_2 = -\mu_2 \) and \( \gamma_1 = \gamma_2 \), the conserved “linear momentum density” is
\[ T^I_m = \frac{1}{2} (\mathcal{I}(q^* q_x) + \mathcal{I}(r^* r_x)). \quad (79) \]
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(iii) If, in addition to the conditions in (ii), \( \eta_1 = \eta_2 = 0 \), the conserved “Hamiltonian density” is

\[
T'_e = -a_1 \mathcal{I}(q^* q_{xxx}) - a_2 \mathcal{I}(r^* r_{xxx}) + b_1 \mathcal{R}(q q^*_{xxx}) + b_2 \mathcal{R}(r r^*_{xxx}) \\
+ \frac{1}{4} c_1 |q|^2 (e_1 |q|^2 - f_1 \mathcal{I}(q^* q_x)) + \frac{1}{4} c_2 |r|^2 (e_2 |r|^2 - f_2 \mathcal{I}(r^* r_x)) \\
+ \frac{1}{2} \delta_1 \mathcal{I}(q^* q_x) + \frac{1}{2} \delta_2 \mathcal{I}(r^* r_x) - \frac{1}{4} \mu_1 |q|^2 \mathcal{I}(q^* q_x) - \frac{1}{4} \mu_2 |r|^2 \mathcal{I}(r^* r_x) \\
+ \frac{1}{2} \gamma_2 |q|^2 |r|^2.
\]

The two-component bright soliton solutions recovered in this paper are structured as

\[
q(x, t) = A_1 \text{sech}^2[B(x - vt)] e^{i(-\kappa x + \omega t + \theta_0)},
\]

\[
r(x, t) = A_2 \text{sech}^2[B(x - vt)] e^{i(-\kappa x + \omega t + \theta_0)}.
\]

Here, \( A_j \) (\( j = 1, 2 \)) are the amplitudes of the two solitons while \( B \) represents their inverse widths. Therefore, the conserved quantities are

\[
P = \frac{1}{2} \int_{-\infty}^{\infty} (|q|^2 + |r|^2) dx = \frac{2}{3B} (|A_1|^2 + |A_2|^2),
\]

\[
M = \frac{1}{4i} \int_{-\infty}^{\infty} \{(q^* q_x - q q^*_x) + (r^* r_x - r r^*_x)\} dx = -\frac{2\kappa}{3B} (|A_1|^2 + |A_2|^2)
\]

and

\[
H = \int_{-\infty}^{\infty} \left[ -\frac{1}{2i} \{a_1 (q^* q_{xxx} - q q^*_xxx) + a_2 (r^* r_{xxx} - r r^*_xxx)\} \\
+ \frac{1}{2} \{b_1 (q^* qxxxx + q q^*_xxxx) + b_2 (r^* rxxxx + r r^*_xxxx)\} \\
- \frac{1}{8i} \{(c_1 f_1 + \mu_1)|q|^2 (q^* q_x - q q^*_x) + (c_2 f_2 + \mu_2)|r|^2 (r^* r_x - r r^*_x)\} \\
+ \frac{1}{4i} \{\delta_1 (q^* q_x - q q^*_x) + \delta_2 (r^* r_x - r r^*_x)\} + \frac{\gamma_2}{2} |q|^2 |r|^2 \right] dx
\]

\[
= \frac{4\kappa}{15} (a_1 A_1^2 + a_2 A_2^2) (5\kappa^2 + 12B^2) + \frac{64}{21} B^5 (b_1 A_1^2 + b_2 A_2^2) \\
+ \frac{224}{35} \kappa^2 B (b_1 A_1^4 + b_2 A_2^4) + \frac{4\kappa^4}{3B} (b_1 A_1^5 + b_2 A_2^5) \\
+ \frac{8\kappa}{35B} \{(c_1 f_1 + \mu_1) A_1^4 + (c_2 f_2 + \mu_2) A_2^4\} - \frac{2\kappa}{3B} (\delta_1 A_1^2 + \delta_2 A_2^2) + \frac{16\gamma_2 A_1^2 A_2^2}{35B}.
\]

These conserved quantities are power (\( P \)), linear momentum (\( M \)) and Hamiltonian (\( H \)), respectively.
6. Conclusions

Today’s work is a display of polarized soliton solutions and their conservation laws with FLE in birefringent fibers. Two integration algorithms have made this retrieval of soliton solutions possible while the approach of multipliers has successfully recovered the conservation laws. The conserved quantities are finally computed and exhibited by utilizing the soliton solutions that are recovered in the paper. These recovered and enumerated results serve as a gateway to extend the study further along in this direction. An immediate example would be to extend the model to address DWDM topology with FLE, studying the cascaded system and Thirring solitons. More results, aligned with recently reported works, will appear over time sequentially. Consequently, the aspects of ghost pulses and suppression of intra-channel collision of solitons, stochastic perturbation of solitons are all on the table. These projects will be taken up and their results would be disseminated.

Disclosure

The authors also declare that there is no conict of interest.

References


