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ELECTRODYNAMICS AND WAVE PROPAGATION

Pure-Cubic Optical Soliton Perturbation with Complex Ginzburg–Landau Equation Having a Dozen Nonlinear Refractive Index Structures

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Abstract—This paper recovers soliton solutions to perturbed pure–cubic complex Ginzburg–Landau equation having a dozen forms of nonlinear refractive index. Two integration schemes, namely the new mapping method and the addendum to Kudryashov's approach have made this retrieval possible. Bright, dark and singular soliton solutions are recovered and enumerated for every nonlinear form. As a byproduct of the schemes, periodic solutions have emerged and are presented as well.

Keywords: solitons, cubic-quartic solitons, Ginzburg–Landau equation

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1. INTRODUCTION

There exists an abundance of models that govern the dynamics of soliton propagation through an optical fiber, PCF, metamaterials or any other kind of waveguides. While the fundamental model is the nonlinear Schrödinger's equation (NLSE) several alternate, but equally popular, models have sprung up with time. A few of them are Radhakrishnan–Kundu–Lakshmanan equation, Lakshmanan–Porsezian–Daniel model, Schrödinger–Hirota equation, Gabitov–Turitsyn equation, Fokas–Lenell's equation, complex Ginzburg–Landau equation (CGLE) [1–18], and many more. These are well known models for polarization–preserving fibers. Similarly, in birefringent fibers, there exists several popular models that study split pulses and a couple of them are Manakov equation and Thirring's model. On the other hand in (2+1)-dimensions, an alternate model to study optical dromions would be the recently proposed Kundu–Mukherjee–Naskar equation. For every such model, and every single situation, the basic governing dynamics for soliton propagation is the existence of a delicate balance between chromatic dispersion (CD) and self–phase modulation (SPM).

A preposterous consequence can occur, during fiber optic pulse transmission, when CD runs low. Many forms of technological backup plans have been proposed and implemented to circumvent this crisis situation. One proposal is the introduction of Bragg gratings with dispersive reflectivity that can compensate this low count of CD. Another approach was the introduction of pure-quartic solitons that came up during 2016 [19]. Here, CD is replaced by fourth–order dispersion (4OD). The drawback for such a model is that it can only be studied numerically and one can only retrieve stationary optical solitons analytically for pure-quartic NLSE. Thus, pure-quartic solitons never gained popularity as an alternative model to address this crisis situation. Subsequently, the concept of cubic–quartic (CQ) solitons have emerged where third–order dispersion (3OD) and 4OD together replaces CD [20–26]. This concept had picked up momentum and a flood of analytical results have started pouring in.

Yet another alternative, but similar, concept has been proposed during 2019 to salvage this crisis situation. The concept of pure-cubic (PC) solitons was introduced during 2019 [27, 28]. Here, CD is replaced only with 3OD unlike CQ solitons. One feature with PC solitons is that it is necessary to include a few Hamiltonian perturbation terms for the governing model to be solvable. NLSE with 3OD, replacing CD, is not rendered to be integrable. Later, several papers have started streaming with this new evolving concept [29–31]. The current paper will study PC solitons with CGLE as its platform. The perturbations terms are all of Hamiltoniantype where the nonlinear ones are considered with maximum intensity. A dozen forms of SPM that stems out of nonlinear refractive index are studied. Two integration schemes are implemented to secure bright, dark and singular soliton solutions to the model. The details are enumerated and the soliton solutions are exhibited in the rest of the paper after a quick intro to the governing model along with two integration algorithms.

1.1. Governing Model

The dimensionless form of PC CGLE with Hamiltonian perturbation terms is written as:

$$\begin{aligned} iq_t + iaq_{xxx} + F\left(|q|^2\right)q &= \alpha\frac{|q_x|^2}{q^*} \\ &+ \frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \\ &+ i\left[\delta q_x + \lambda\left(|q|^{2m}q\right)_x + \mu\left(|q|^{2m}\right)_x q + \nu|q|^{2m}q_x\right], \end{aligned} \tag{1}$$

provided $|q| \neq 0$. Here a, α, β, γ, δ, λ, μ and ν are real–valued constants, while m represents parameter of maximum intensity with $i = \sqrt{-1}$. The first term represents linear temporal evolution, a is the coefficient of 3OD effects, the functional F represents a general form of the intensity dependent refractive index. The terms due to α, β and γ are from the perturbation effects; in particular, γ comes from the detuning effect. Next, δ is the coefficient of inter–modal dispersions (IMD) and the parameter λ gives the self-steepening (SS) term that avoid the formation of shock waves. Finally μ and ν stand for the coefficients of nonlinear dispersion. Here, the dependent variable $q = q(x,t)$ represents the complex-valued wave profile, while $q^*(x,t)$ is its conjugate, while x represents the non-dimensional distance along the fiber and t represents the time in dimensionless form. The functional F gives the range of nonlinear refractive index structures.

Equation (1) is a manifested version of the standard model that governs the propagation of soliton molecules through optical fibers across trans-oceanic and trans-continental distances. This is the well known NLSE, with CD, that is structured as:

$$iq_t + aq_{xx} + F\left(|q|^2\right)q = 0, \tag{2}$$

where a is the coefficient of CD. In (1) for CGLE, it is this CD that is replaced by 3OD which therefore formulates the dispersive effect.

The objective of this article is to locate solitons and other solutions of Eq. (1) using two integration algorithms declared earlier. It needs to be pointed out that Eq. (1) has been studied recently, for the special case $\alpha = \beta = \gamma = 0$, using the method of unified Riccati equation expansion [30, 31].

This article is organized as follows: In Sections 2 and 3, the new mapping method and the addendum to Kudryashov's method are revisited. In Section 4, the mathematical analysis of Eq. (1) is displayed. In Sections 5–16, Eq. (1) is solved for twelve forms of SPM using two algorithms as recapitulated in Sections 2 and 3. To close, Section 17 sums up the paper with a few conclusive lines along with potential new avenues to expand and move further along.

2. NEW MAPPING METHOD

Suppose that we have the following nonlinear evolution equation (NLEE):

$$P(q, q_x, q_t, q_{xx}, q_{tt}, \ldots) = 0, \tag{3}$$

where $q = q(x,t)$ is an unknown function, P is a polynomial in $q(x,t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. The main steps of this method can be summarized as follows:

Step 1. We use the traveling wave transformation

$$\begin{gathered} q(x,t) = U(\xi), \\ \xi = \kappa x - \omega t, \end{gathered} \tag{4}$$

where κ and ω are nonzero constants, to reduce Eq. (3) into the following nonlinear ordinary differential equation (ODE):

$$P(U, U', U'', \ldots) = 0, \tag{5}$$

where P is a polynomial in $U(\xi)$ and its total derivatives, such that $' = \frac{d}{d\xi}$.

Step 2. We assume that Eq. (5) has the formal solution

$$U(\xi) = \sum_{l=0}^{2N} \delta_l F^l(\xi), \tag{6}$$

where $F(\xi)$ satisfies the first order auxiliary ODE:

$$F'^2(\xi) = r + pF^2(\xi) + \frac{1}{2}hF^4(\xi) + \frac{1}{3}sF^6(\xi), \tag{7}$$

where δ_l $(l = 0, \ldots, 2N)$, r, p, h and s are arbitrary constants to be determined, such that $s \neq 0$.

Step 3. We determine the balance number N of (6) by balancing the highest nonlinear terms and the highest order derivatives of $U(\xi)$ in Eq. (5).

Step 4. We substitute (6) along with (7) into Eq. (5), collect all the coefficients of $F^{m_1}(F')^j$ $(m_1 = 0,1,2,\ldots)$ and $(j = 0,1)$ and set them to zero, to get a system of algebraic equations for δ_l $(l = 0,1,\ldots,2N), r, p, h, s, \kappa$ and ω.

Step 5. We solve the system of algebraic equations obtained in Step 4, using the Maple, to find δ_l $(l = 0,1,\ldots,2N), r, p, h, s, \kappa$ and ω.

Step 6. It is well known that Eq. (7) admits the following types of solutions with:

Type 1. If $s = \frac{3h_2^2}{16p}$, $r = \frac{16p^2}{27h_2}$, then one recovers:

(1) Soliton solutions:

$$F(\xi) = 4\sqrt{\frac{-p\tanh^2\left(\epsilon\sqrt{-\frac{p}{3}}\xi\right)}{3h\left(3+\tanh^2\left(\epsilon\sqrt{-\frac{p}{3}}\xi\right)\right)}}, \tag{8}$$

and

$$F(\xi) = 4\sqrt{\frac{-p\coth^2\left(\epsilon\sqrt{-\frac{p}{3}}\xi\right)}{3h\left(3+\coth^2\epsilon\left(\varepsilon\sqrt{-\frac{p}{3}}\xi\right)\right)}}, \tag{9}$$

provided

$$p < 0, \quad h > 0. \tag{10}$$

(2) Periodic solutions:

$$F(\xi) = 4\sqrt{\frac{p\tan^2\left(\epsilon\sqrt{\frac{p}{3}}\xi\right)}{3h\left(3-\tan^2\left(\epsilon\sqrt{\frac{p}{3}}\xi\right)\right)}}, \tag{11}$$

and

$$F(\xi) = 4\sqrt{\frac{p\cot^2\left(\epsilon\sqrt{\frac{p}{3}}\xi\right)}{3h\left(3-\cot^2\left(\epsilon\sqrt{\frac{p}{3}}\xi\right)\right)}}, \tag{12}$$

provided

$$p > 0, \quad h < 0. \tag{13}$$

Type 2. If $s = \frac{3h_2^2}{16p}$, $r = 0$, then one recovers dark soliton solution:

$$F(\xi) = \sqrt{\frac{-2p}{h}\left(1+\tanh\left(\epsilon\sqrt{p}\xi\right)\right)}, \quad p > 0, \tag{14}$$

and singular solitons

$$F(\xi) = \sqrt{\frac{-2p}{h}\left(1+\coth\left(\epsilon\sqrt{p}\xi\right)\right)}, \quad p > 0. \tag{15}$$

Type 3. If $r = 0$, then one gets the following solutions:

(1) Soliton solutions:

$$F(\xi) = \sqrt{\frac{-6ph\operatorname{sech}^2\left(\sqrt{p}\xi\right)}{3h^2 - 4ps\left(1+\epsilon\tanh\left(\sqrt{p}\xi\right)\right)^2}}, \quad p > 0, \tag{16}$$

$$F(\xi) = \sqrt{\frac{6ph\operatorname{cosech}^2\left(\sqrt{p}\xi\right)}{3h^2 - 4ps\left(1+\epsilon\coth\left(\sqrt{p}\xi\right)\right)^2}}, \quad p > 0, \tag{17}$$

$$F(\xi) = \sqrt{\frac{-6p\operatorname{sech}^2\left(\sqrt{p}\xi\right)}{3h + 4\epsilon\sqrt{3ps}\tanh\left(\sqrt{p}\xi\right)}}, \quad p > 0, \quad s > 0, \tag{18}$$

$$F(\xi) = \sqrt{\frac{6p\operatorname{cosech}^2\left(\sqrt{p}\xi\right)}{3h + 4\epsilon\sqrt{3ps}\coth\left(\sqrt{p}\xi\right)}}, \quad p > 0, \quad s > 0. \tag{19}$$

(2) Bright soliton solutions:

$$F(\xi) = 2\sqrt{\frac{3p\epsilon\operatorname{sech}\left(2\sqrt{p}\xi\right)}{\sqrt{M} - 3\epsilon h\operatorname{sech}\left(2\sqrt{p}\xi\right)}}, \quad p > 0, \quad M > 0, \tag{20}$$

$$F(\xi) = 2\sqrt{\frac{3p\operatorname{sech}^2\left(\epsilon\sqrt{p}\xi\right)}{2\sqrt{M} - \left(\sqrt{M}+3h\right)\operatorname{sech}^2\left(\epsilon\sqrt{p}\xi\right)}}, \quad p > 0, \quad h < 0, \quad s < 0, \quad M > 0. \tag{21}$$

(3) Singular soliton solutions:

$$F(\xi) = 2\sqrt{\frac{3p\operatorname{cosech}^2\left(\epsilon\sqrt{p}\xi\right)}{2\sqrt{M} + \left(\sqrt{M}-3h\right)\operatorname{cosech}^2\left(\epsilon\sqrt{p}\xi\right)}}, \quad p > 0, \quad h < 0, \quad s < 0, \quad M > 0. \tag{22}$$

$$F(\xi) = 2\sqrt{\frac{3\epsilon p\operatorname{cosech}\left(2\sqrt{p}\xi\right)}{\sqrt{-M} - 3\epsilon h\operatorname{cosech}\left(2\sqrt{p}\xi\right)}}, \quad p > 0, \quad M < 0. \tag{23}$$

(4) The periodic solutions:

$$F(\xi) = \sqrt{\frac{-6p\sec^2\left(\sqrt{-p}\xi\right)}{3h + 4\epsilon\sqrt{-3ps}\tan\left(\sqrt{-p}\xi\right)}}, \quad p < 0, \quad s > 0, \tag{24}$$

$$F(\xi)=\sqrt{\frac{-6p\operatorname{cosec}^2\left(\sqrt{-p}\xi\right)}{3h+4\epsilon\sqrt{-3ps}\cot\left(\sqrt{-p}\xi\right)}},\qquad(25)$$
$$p<0,\quad s>0,$$

$$F(\xi)=2\sqrt{\frac{-3p\sec^2\left(\epsilon\sqrt{-p}\xi\right)}{2\sqrt{M}-\left(\sqrt{M}-3h\right)\sec^2\left(\epsilon\sqrt{-p}\xi\right)}},\qquad(26)$$
$$p<0,\quad h>0,\quad s<0,\quad M>0,$$

$$F(\xi)=2\sqrt{\frac{3p\operatorname{cosec}^2\left(\epsilon\sqrt{-p}\xi\right)}{2\sqrt{M}-\left(\sqrt{M}+3h\right)\operatorname{cosec}^2\left(\epsilon\sqrt{-p}\xi\right)}},\qquad(27)$$
$$p<0,\quad h>0,\quad s<0,\quad M>0,$$

$$F(\xi)=2\sqrt{\frac{3\epsilon p\sec\left(2\sqrt{-p}\xi\right)}{\sqrt{M}-3\epsilon h\sec\left(2\sqrt{-p}\xi\right)}},\qquad(28)$$
$$p<0,\quad M>0,$$

$$F(\xi)=2\sqrt{\frac{3\epsilon p\operatorname{cosec}\left(2\sqrt{-p}\xi\right)}{\sqrt{M}-3\epsilon h\operatorname{cosec}\left(2\sqrt{-p}\xi\right)}}.\qquad(29)$$
$$p<0,\quad M>0.$$

Here

$$M=9h^2-48ps\ \text{and}\ \epsilon=\pm1.\qquad(30)$$

Step 7. We substitute the values δ_l $(l=0,1,\ldots,2N)$, p, h, s, κ and ω as well as the solutions (8)–(30) into (6) to get hyperbolic and periodic function solutions of Eq. (5).

3. ADDENDUM TO KUDRYASHOV'S METHOD

Recently, Kudryashov proposed a new method for solving NLEEs. Based on this new Kudryashov's method, we will describe in this section the addendum Kudryashov's method. The main steps of this method can be summarized as follows:

Step 1. We assume that Eq. (5) has the formal solution

$$U(\xi)=\sum_{g=0}^{N}\beta_g R^g(\xi),\qquad(31)$$

where β_g $(g=0,1,2,\ldots,N)$ are constants to be determined, such that $\beta_N\neq0$, while $R(\xi)$ satisfies the auxiliary ODE:

$$R'^2(\xi)=R^2(\xi)\left[1-\chi R^{2p}(\xi)\right]\ln^2 k,\quad 0<k\neq1,\qquad(32)$$

where χ is a constant. It is easy to show that Eq. (32) has the following solutions:

$$R(\xi)=\left[\frac{4A}{4A^2\exp_k(p\xi)+\chi\exp_k(-p\xi)}\right]^{\frac{1}{p}},\qquad(33)$$

where A is a nonzero constant, p is a positive integer and $\exp_k(p\xi)=k^{p\xi}$.

Step 2. We determine the relationship between N and p as follows:

If $D[U(\xi)]=N$, $D[U'(\xi)]=N+p$, $D[U''(\xi)]=N+2p$, consequently we have $D\left[U^r(\xi)U^{(s)}(\xi)\right]=N(r+1)+sp$.

Step 3. We substitute (31) along with Eq. (32) into Eq. (5), equate all the coefficients of $[R(\xi)]^{m_1}[R'(\xi)]^j$, $(m_1=0,1,2,\ldots,j=0,1)$ to zero, yield a set of algebraic equations which can be solved by Maple to find the values of β_g $(g=0,1,2,\ldots,N)$ and c. Consequently, we can get the exact solutions of Eq. (5).

Note: If we set $k=e$ (i.e. $\ln e=1$) and $p=1$, then this method reduces to the new Kudryashov's method reported earlier.

4. MATHEMATICAL ANALYSIS

For the traveling wave solutions of Eq. (1), the starting hypothesis is taken to be

$$q(x,t)=\phi(\xi)\exp[i\psi(x,t)],\qquad(34)$$

where $\phi(\xi)$ and $\psi(x,t)$ are real functions, such that

$$\xi=x-ct,\quad \psi(x,t)=-\kappa x+\omega t+\theta_0,\qquad(35)$$

and c,κ,ω and θ_0 are real constants. Here $\phi(\xi)$ represents the pulse shape which is a real function, c is the velocity of the soliton, κ is the soliton frequency, ω is the soliton wave number and θ_0 is a phase constant. Substituting (34) along with (35) into Eq. (1) and separating the real and imaginary parts, one gets the real part in the form:

$$\begin{aligned}(3a\kappa-\beta)\phi\phi''-\alpha\phi'^2-\left(\omega+a\kappa^3+\alpha\kappa^2+\delta\kappa+\gamma\right)\phi^2\\+F\left(\phi^2\right)\phi^2-\kappa(\lambda+\nu)\phi^{2m+2}=0,\end{aligned}\qquad(36)$$

and the imaginary part in the form:

$$\begin{aligned}a\phi'''-\left(c+\delta+3ak^2\right)\phi'\\-[(2m+1)\lambda+2m\mu+\nu]\phi^{2m}\phi'=0.\end{aligned}\qquad(37)$$

By integrating of (37), one gets:

$$\begin{aligned}a\phi''-\left(c+\delta+3ak^2\right)\phi\\-\frac{1}{2m+1}[(2m+1)\lambda+2m\mu+\nu]\phi^{2m+1}=0.\end{aligned}\qquad(38)$$

Integration of (38) yields to

$$\phi(\xi) = \exp\left(-\sqrt{\frac{c+\delta+3ak^2}{a}}\xi\right), \tag{39}$$

that leads to the velocity c of the soliton, under the constraint conditions

$$\begin{gathered}(2m+1)\lambda + 2m\mu + \nu = 0,\\ a\left(c+\delta+ak^2\right) > 0.\end{gathered} \tag{40}$$

Equation (39) indicates that the soliton exists if and only if the conditions (40) are satisfied.

Now, Eq. (36) can be rewritten as:

$$\begin{gathered}\Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2\\ + F\left(\phi^2\right)\phi^2 - \kappa(\lambda+\nu)\phi^{2m+2} = 0,\end{gathered} \tag{41}$$

where

$$\begin{gathered}\Delta_0 = 3a\kappa - \beta,\\ \Delta_2 = \omega + a\kappa^3 + \alpha\kappa^2 + \delta\kappa + \gamma.\end{gathered} \tag{42}$$

The task now is to solve Eq. (41) using the above two methods when the functional $F\left(\phi^2\right)$ takes the following forms.

5. KERR LAW

For the Kerr law nonlinearity, we have

$$F(\phi) = b\phi, \tag{43}$$

where b is a nonzero constant. The Kerr law of nonlinearity originates from the fact that a light wave in an optical fiber faces nonlinear responses. Even though the nonlinear responses are extremely weak, their effects appear in various ways over long distance of propagation that is measured in terms of light wavelength. The origin of nonlinear response is related to the non-harmonic motion of bound electrons under the influence of an applied field.

Equation (1) corresponding to Kerr law nonlinearity (43) is given by:

$$\begin{gathered}iq_t + iaq_{xxx} + b|q|^2q = \alpha\frac{|q_x|^2}{q^*}\\ + \frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q\\ + i\left[\delta q_x + \lambda\left(|q|^{2m}q\right)_x + \mu\left(|q|^{2m}\right)_x q + \nu|q|^{2m}q_x\right],\end{gathered} \tag{44}$$

where Eq. (41) reduces to:

$$\begin{gathered}\Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2\\ + b\phi^4 - \kappa(\lambda+\nu)\phi^{2m+2} = 0.\end{gathered} \tag{45}$$

For integrability, one must select $m = 1$. This leads to the modification of Eq. (1) corresponding to Kerr law nonlinearity as:

$$\begin{gathered}iq_t + iaq_{xxx} + b|q|^2q = \alpha\frac{|q_x|^2}{q^*}\\ + \frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q\\ + i\left[\delta q_x + \lambda\left(|q|^2q\right)_x + \mu\left(|q|^2\right)_x q + \nu|q|^2q_x\right].\end{gathered} \tag{46}$$

Consequently, Eq. (45) becomes:

$$\begin{gathered}\Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2\\ + [b - \kappa(\lambda+\nu)]\phi^4 = 0.\end{gathered} \tag{47}$$

In the next two subsections, we will solve Eq. (47) using the following two methods:

5.1. New Mapping Method

According to the new mapping method, we balance $\phi\phi''$ with ϕ^4 in Eq. (47) yields the balance number $N = 1$. Now, from (6), the solution of Eq. (47) has the form:

$$\phi(\xi) = \delta_0 + \delta_1F(\xi) + \delta_2F^2(\xi), \tag{48}$$

where $\delta_0, \delta_1, \delta_2$ are constants to be determined, such that $\delta_2 \neq 0$, while $F(\xi)$ satisfies the first order nonlinear auxiliary ODE (7). Substituting (48) along with (7) into Eq. (47), collecting all the coefficients of $F^l(\xi)[F'(\xi)]^j$ $(l = 0,1,\ldots 8,\ j = 0,1)$ and setting them to zero, we have the following algebraic equations:

$$\begin{gathered}3[b-\kappa(\lambda+\nu)]\delta_2^4 - 4\alpha\delta_2^2s + 8\Delta_0\delta_2^2s = 0,\\ 12[b-\kappa(\lambda+\nu)]\delta_1\delta_2^3 + 11\Delta_0\delta_1\delta_2s - 4\alpha\delta_1\delta_2s = 0,\\ 6[b-\kappa(\lambda+\nu)]\delta_1^2\delta_2^2 + \frac{8}{3}\Delta_0\delta_0\delta_2s\\ + \Delta_0\delta_1^2s - 2\alpha\delta_2^2h - \frac{1}{3}\alpha\delta_1^2s\\ + 3\Delta_0\delta_2^2h + 4[b-\kappa(\lambda+\nu)]\delta_0\delta_2^3 = 0,\\ 4\Delta_0\delta_1\delta_2h + 4\left\{3\delta_0\delta_1\delta_2^2 + \delta_1^3\delta_2\right\}[b-\kappa(\lambda+\nu)]\\ + \Delta_0\delta_0\delta_1s - 2\alpha\delta_1\delta_2h = 0,\\ \Delta_0\delta_1^2h + 3\Delta_0\delta_0\delta_2h - \Delta_2\delta_2^2\\ + \left\{12\delta_0\delta_1^2\delta_2 + \delta_1^4\right\}[b-\kappa(\lambda+\nu)] - \frac{1}{2}\alpha\delta_1^2h\\ + 4\Delta_0\delta_2^2p - 4\alpha\delta_2^2p + 6[b-\kappa(\lambda+\nu)]\delta_0^2\delta_2^2 = 0,\\ 5\Delta_0\delta_1\delta_2p + \Delta_0\delta_0\delta_1h\\ + 12[b-\kappa(\lambda+\nu)]\delta_0^2\delta_1\delta_2 - 4\alpha\delta_1\delta_2p\\ + 4[b-\kappa(\lambda+\nu)]\delta_0\delta_1^3 - 2\Delta_2\delta_1\delta_2 = 0,\end{gathered} \tag{49}$$

$$-\Delta_2\delta_1^2-4\alpha\delta_2^2 r+\Delta_0\delta_1^2 p+4[b-\kappa(\lambda+\nu)]\delta_0^3\delta_2$$
$$-\alpha\delta_1^2 p+6[b-\kappa(\lambda+\nu)]\delta_0^2\delta_1^2$$
$$-2\Delta_2\delta_0\delta_2+2\Delta_0\delta_2^2 r+4\Delta_0\delta_0\delta_2 p=0,$$
$$4[b-\kappa(\lambda+\nu)]\delta_0^3\delta_1+2\Delta_0\delta_1\delta_2 r$$
$$+\Delta_0\delta_0\delta_1 p-2\Delta_2\delta_0\delta_1-4\alpha\delta_1\delta_2 r=0,$$
$$2\Delta_0\delta_0\delta_2 r-\alpha r\delta_1^2+[b-\kappa(\lambda+\nu)]\delta_0^4-\Delta_2\delta_0^2=0.$$

With the aid of the solutions (8)–(29), we have the following types of solutions:

Type 1. Substituting $s=\frac{3h^2}{16p}$, $r=\frac{16p^2}{27h}$, into the algebraic equations (49) and solve them by Maple, one gets the following results:

$$\delta_0=\epsilon\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}},$$
$$\delta_1=0,\quad \delta_2=\frac{\epsilon h\Delta_0}{2\Delta_2}\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}}, \tag{50}$$
$$p=\frac{3\Delta_2}{2\Delta_0},\quad h=h,\quad \alpha=\frac{3}{2}\Delta_0,$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]<0$ and $\epsilon=\pm1$.

If we substitute (50) along with (8)–(12) into Eq. (48), then Eq. (46) has the following solutions:

5.1.1. Soliton solutions.

$$q(x,t)=\epsilon\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}}\times\left[1-\frac{4\tanh^2\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}{3+\tanh^2\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{51}$$

and

$$q(x,t)=\epsilon\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}}\times\left[1-\frac{4\coth^2\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}{3+\coth^2\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{52}$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]<0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

5.1.2. Periodic solutions.

$$q(x,t)=\epsilon\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}}\times\left[1+\frac{4\tan^2\left(\epsilon\sqrt{\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}{3-\tan^2\left(\epsilon\sqrt{\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{53}$$

and

$$q(x,t)=\epsilon\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}}\times\left[1+\frac{4\cot^2\left(\epsilon\sqrt{\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}{3-\cot^2\left(\epsilon\sqrt{\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{54}$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]<0$, $\Delta_0\Delta_2>0$ and $\epsilon=\pm1$.

Type 2. Substituting $s=\frac{3h^2}{16p}$, $r=0$, into the algebraic equations (49) and solve them by Maple, we get the following results:

$$\delta_0=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}},$$
$$\delta_1=0,\quad \delta_2=-\frac{\epsilon h\Delta_0}{2\Delta_2}\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}, \tag{55}$$
$$p=-\frac{\Delta_2}{2\Delta_0},\quad h=h,\quad \alpha=\frac{3}{2}\Delta_0.$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$ and $\epsilon=\pm1$.

If we substitute (55) along with (14) and (15) into Eq. (48), then Eq. (46) has the following solutions.

5.1.3. Dark and singular solitons.

$$q(x,t)=\frac{\epsilon}{2}\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1+\tanh\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{56}$$

and

$$q(x,t)=\frac{\epsilon}{2}\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1+\coth\left(\varepsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{57}$$

respectively, provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

Type 3. Substituting $r=0$, into the algebraic equations (49) and solve them by Maple, we get the following results:

$$\delta_0=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}},$$
$$\delta_1=0,\quad \delta_2=-\frac{3\epsilon h\Delta_0}{4\Delta_2}\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}, \tag{58}$$
$$p=-\frac{\Delta_2}{2\Delta_0},\quad s=-\frac{9h^2\Delta_0}{32\Delta_2},\quad h=h,\quad \alpha=\frac{1}{2}\Delta_0,$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$ and $\epsilon=\pm1$. If we substitute (58) along with (16)–(29) into Eq. (48), then Eq. (46) has the following solutions.

5.1.4. Soliton solution.

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1-\frac{12\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}{16-3\left[1+\tanh\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)\right]^2}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (59)$$

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1+\frac{12\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}{16-3\left[1+\coth\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)\right]^2}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (60)$$

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1-\frac{3\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}{4+2\sqrt{3}\tanh\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}. \quad (61)$$

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1+\frac{3\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}{4+2\sqrt{3}\coth\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (62)$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

5.1.5. Bright soliton.

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{[b-\kappa(\lambda+\nu)]}}\times\left[1+\frac{3}{\cosh\left(\epsilon\sqrt{-\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)-2}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (63)$$

and

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1+\frac{3}{2\cosh^2\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)-3}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (64)$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

5.1.6. Singular soliton.

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1-\frac{3}{2\sinh^2\left(\epsilon\sqrt{-\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)+3}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (65)$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

5.1.7. Periodic solutions.

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1+\frac{3\sec^2\left(\epsilon\sqrt{\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}{2-3\sec^2\left(\epsilon\sqrt{\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (66)$$

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1+\frac{3\operatorname{cosec}^2\left(\epsilon\sqrt{\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}{2-3\operatorname{cosec}^2\left(\epsilon\sqrt{\frac{\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (67)$$

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1+\frac{3\sec\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}{1-2\sec\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (68)$$

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1+\frac{3\operatorname{cosec}\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}{1-2\operatorname{cosec}\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (69)$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2>0$ and $\epsilon=\pm1$.

5.2. Addendum to Kudryashov's Method

According to this method, we balance $\phi\phi''$ with ϕ^4 in Eq. (47), one gets the relation:

$$2N+2p=4N\Rightarrow N=p. \quad (70)$$

Now, we will discuss the following cases:

Case 1. If we choose $p=1$, then $N=1$. Thus, we deduce that from (31) that Eq. (47) has the formal solution:

$$\phi(\xi)=\beta_0+\beta_1 R(\xi), \tag{71}$$

where β_0 and β_1 are constants to be determined, such that $\beta_1 \neq 0$ and the function $R(\xi)$ satisfies the auxiliary ODE (32). Substituting (71) along with (32) into Eq. (47), collecting all the coefficients of each power of $[R(\xi)]^{m_1}\,[R'(\xi)]^{j}$, $(m_1=0,1,2,\ldots,4,\ j=0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
\alpha\beta_1^2\chi\ln^2 k-2\Delta_0\beta_1^2\chi\ln^2 k+[b-\kappa(\lambda+\nu)]\beta_1^4=0,\\
4[b-\kappa(\lambda+\nu)]\beta_0\beta_1^3-2\Delta_0\beta_1\beta_0\chi\ln^2 k=0,\\
6[b-\kappa(\lambda+\nu)]\beta_0^2\beta_1^2\\
-\alpha\beta_1^2\ln^2 k-\Delta_2\beta_1^2+\Delta_0\beta_1^2\ln^2 k=0,\\
\Delta_0\beta_1\beta_0\ln^2 k-2\Delta_2\beta_0\beta_1+4[b-\kappa(\lambda+\nu)]\beta_0^3\beta_1=0,\\
[b-\kappa(\lambda+\nu)]\beta_0^4-\Delta_2\beta_0^2=0.
\end{gathered} \tag{72}$$

On solving the above algebraic Eqs. (72) by using the Maple, one gets the following results:

$$\begin{gathered}
\beta_0=0,\quad \beta_1=\epsilon\sqrt{\frac{\left(\Delta_2+\Delta_0\ln^2 k\right)\chi}{b-\kappa(\lambda+\nu)}},\\
\alpha=\frac{-\Delta_2+\Delta_0\ln^2 k}{\ln^2 k},
\end{gathered} \tag{73}$$

provided $\left(\Delta_2+\Delta_0\ln^2 k\right)[b-\kappa(\lambda+\nu)]\chi>0$ and $\epsilon=\pm 1$.

Substituting (73) along with (33) into Eq. (71), one gets the solutions of Eq. (46) in the form:

$$\begin{gathered}
q(x,t)=\epsilon\sqrt{\frac{\left(\Delta_2+\Delta_0\ln^2 k\right)\chi}{b-\kappa(\lambda+\nu)}}\\
\times\left[\frac{4A}{4A^2\exp_k[(x-ct)]+\chi\exp_k[-(x-ct)]}\right]e^{i(-\kappa x+\omega t+\theta_0)}.
\end{gathered} \tag{74}$$

In particular, if we set $\chi=4A^2$ in (74), then we have the bright soliton solution of Eq. (46) as:

$$\begin{gathered}
q(x,t)=\epsilon\sqrt{\frac{\Delta_2+\Delta_0\ln^2 k}{b-\kappa(\lambda+\nu)}}\\
\times\operatorname{sech}[(x-ct)\ln k]e^{i(-\kappa x+\omega t+\theta_0)},
\end{gathered} \tag{75}$$

while, if $\chi=-4A^2$, one gets the singular soliton solution of Eq. (46) as:

$$\begin{gathered}
q(x,t)=\epsilon\sqrt{-\frac{\Delta_2+\Delta_0\ln^2 k}{b-\kappa(\lambda+\nu)}}\\
\times\operatorname{cosech}[(x-ct)\ln k]e^{i(-\kappa x+\omega t+\theta_0)}.
\end{gathered} \tag{76}$$

Case 2. If we choose $p=2$, then $N=2$. Thus, we deduce that from (31) that Eq. (46) has the formal solution:

$$\phi(\xi)=\beta_0+\beta_1 R(\xi)+\beta_2 R^2(\xi), \tag{77}$$

where β_0,β_1 and β_2 are constants to be determined, such that $\beta_2\neq 0$ and the function $R(\xi)$ satisfies the auxiliary ODE (32). Substituting (77) along with Eq. (32) into Eq. (47), collecting all the coefficients of each power of $[R(\xi)]^{m_2}\,[R'(\xi)]^{j}$, $(m_2=0,1,2,\ldots 8,\ j=0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
[b-\kappa(\lambda+\nu)]\beta_2^4-8\Delta_0\beta_2^2\chi\ln^2 k+4\alpha\chi\beta_2^2\ln^2 k=0,\\
4[b-\kappa(\lambda+\nu)]\beta_1\beta_2^3\\
+4\alpha\chi\beta_1\beta_2\ln^2 k-11\Delta_0\beta_1\beta_2\chi\ln^2 k=0,\\
\alpha\beta_1^2\chi\ln^2 k+4[b-\kappa(\lambda+\nu)]\beta_0\beta_2^3\\
-3\Delta_0\beta_1^2\chi\ln^2 k-8\Delta_0\beta_0\beta_2\chi\ln^2 k\\
+6[b-\kappa(\lambda+\nu)]\beta_1^2\beta_2^2=0,\\
-3\Delta_0\beta_1\beta_0\chi\ln^2 k+12[b-\kappa(\lambda+\nu)]\beta_0\beta_1\beta_2^2\\
+4[b-\kappa(\lambda+\nu)]\beta_1^3\beta_2=0,\\
-\Delta_2\beta_2^2+6[b-\kappa(\lambda+\nu)]\beta_0^2\beta_2^2\\
+[b-\kappa(\lambda+\nu)]\beta_1^4-4\alpha\beta_2^2\ln^2 k+4\Delta_0\beta_2^2\ln^2 k\\
+12[b-\kappa(\lambda+\nu)]\beta_0\beta_1^2\beta_2=0,\\
4[b-\kappa(\lambda+\nu)]\beta_0\beta_1^3-4\alpha\beta_1\beta_2\ln^2 k\\
+5\Delta_0\beta_1\beta_2\ln^2 k-2\Delta_2\beta_1\beta_2\\
+12[b-\kappa(\lambda+\nu)]\beta_0^2\beta_1\beta_2=0,\\
4\Delta_0\beta_0\beta_2\ln^2 k-\alpha\beta_1^2\ln^2 k\\
+\Delta_0\beta_1^2\ln^2 k+4[b-\kappa(\lambda+\nu)]\beta_0^3\beta_2\\
+6[b-\kappa(\lambda+\nu)]\beta_0^2\beta_1^2-2\Delta_2\beta_0\beta_2-\Delta_2\beta_1^2=0,\\
\Delta_0\beta_1\beta_0\ln^2 k-2\Delta_2\beta_0\beta_1+4[b-\kappa(\lambda+\nu)]\beta_0^3\beta_1=0,\\
-\Delta_2\beta_0^2+[b-\kappa(\lambda+\nu)]\beta_0^4=0.
\end{gathered} \tag{78}$$

On solving the above algebraic Eqs. (78) by using the Maple, one gets the following results:

$$\begin{gathered}
\beta_0=0,\quad \beta_1=0,\\
\beta_2=\epsilon\sqrt{\frac{\left(\Delta_2+4\Delta_0\ln^2 k\right)\chi}{b-\kappa(\lambda+\nu)}},\\
\alpha=\frac{-\Delta_2+4\Delta_0\ln^2 k}{4\ln^2 k},
\end{gathered} \tag{79}$$

provided $\left(\Delta_2+4\Delta_0\ln^2 k\right)[b-\kappa(\lambda+\nu)]\chi>0$ and $\epsilon=\pm 1$.

Substituting (79) along with (33) into Eq. (77), one gets the solutions of Eq. (46) in the form:

$$q(x,t)=\epsilon\sqrt{\frac{\left(\Delta_2+4\Delta_0\ln^2 k\right)\chi}{b-\kappa(\lambda+\nu)}}\times\left[\frac{4A}{4A^2\exp_k\left[2(x-ct)\right]+\chi\exp_k\left[-2(x-ct)\right]}\right]\times e^{i(-\kappa x+\omega t+\theta_0)}. \quad (80)$$

In particular, if we set $\chi=4A^2$ in (80), then we have the bright soliton solution of Eq. (46) as:

$$q(x,t)=\pm\sqrt{\frac{\Delta_2+4\Delta_0\ln^2 k}{b-\kappa(\lambda+\nu)}}\times\operatorname{sech}\left[2(x-ct)\ln k\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (81)$$

while, if $\chi=-4A^2$, one gets the singular soliton solution of Eq. (46) as:

$$q(x,t)=\pm\sqrt{-\frac{\Delta_2+4\Delta_0\ln^2 k}{b-\kappa(\lambda+\nu)}}\times\operatorname{cosech}\left[2(x-ct)\ln k\right]e^{i(-\kappa x+\omega t+\theta_0)}. \quad (82)$$

Similarly, we can find many other solutions by choosing other values for p and N.

6. POWER LAW

For the power law nonlinearity, we have

$$F(\phi)=b\phi^n, \quad (83)$$

where b is a nonzero constant. Such a situation is commonly visible in nonlinear plasma that addresses the problem of small K-condensation with weak turbulence theory. In this case, it is necessary to emphasize $0<n<2$ for preventing collapse of waves and in particular $n\neq 2$ in order to avoid self-focusing singularity.

Equation (1) corresponding to power law nonlinearity (83) is given by:

$$iq_t+iaq_{xxx}+b|q|^{2n}q=\alpha\frac{|q_x|^2}{q^*}+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_x q+\nu|q|^{2m}q_x\right], \quad (84)$$

where Eq. (41) reduces to:

$$\Delta_0\phi\phi''-\alpha\phi'^2-\Delta_2\phi^2+b\phi^{2n+2}-\kappa(\lambda+\nu)\phi^{2m+2}=0. \quad (85)$$

For integrability, one must select $n=m$. This leads to the modification of Eq. (1) corresponding to power law nonlinearity as:

$$iq_t+iaq_{xxx}+b|q|^{2m}q=\alpha\frac{|q_x|^2}{q^*}+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_x q+\nu|q|^{2m}q_x\right]. \quad (86)$$

Consequently, Eq. (85) changes to:

$$\Delta_0\phi\phi''-\alpha\phi'^2-\Delta_2\phi^2+\left[b-\kappa(\lambda+\nu)\right]\phi^{2m+2}=0. \quad (87)$$

Balancing $\phi\phi''$ and ϕ^{2m+2} in Eq. (87), gives the balance number $N=\frac{1}{m}$. Since the balance number is not integer, then we take into consideration the transformation

$$\phi(\xi)=\left[U(\xi)\right]^{\frac{1}{m}}, \quad (88)$$

where $U(\xi)$ is a new positive function of ξ. Substituting (88) into (87), we have the new equation

$$m\Delta_0UU''-\left[\alpha+(m-1)\Delta_0\right]U'^2-m^2\Delta_2U^2+m^2\left[b-\kappa(\lambda+\nu)\right]U^4=0. \quad (89)$$

In the next subsections, we will solve Eq. (89) using the following two methods.

6.1. New Mapping Method

According to the new mapping method, we balance UU'' with U^4 in Eq. (89) yields the balance number $N=1$. Now, from (6), the solution of Eq. (89) has the form

$$U(\xi)=\delta_0+\delta_1F(\xi)+\delta_2F^2(\xi), \quad (90)$$

where $\delta_0,\delta_1,\delta_2$ are constants to be determined, such that $\delta_2\neq 0$, while $F(\xi)$ satisfies the first order nonlinear auxiliary ODE (7). Substituting (90) along with (7) into Eq. (89), collecting all the coefficients of $F^l(\xi)[F'(\xi)]^j$ $(l=0,1,\dots 8,\ j=0,1)$ and setting them to zero, we have the following algebraic equations:

$$\begin{gathered}
-3m^2\delta_2^4\left[b-\kappa(\lambda+\nu)\right]+4\alpha s\delta_2^2-4\Delta_0m\delta_2^2s-4\Delta_0s\delta_2^2=0,\\
-7\Delta_0m\delta_1\delta_2s-4\Delta_0s\delta_1\delta_2-12m^2\delta_1\delta_2^3\left[b-\kappa(\lambda+\nu)\right]+4\alpha s\delta_1\delta_2=0,\\
-4m^2\delta_0\delta_2^3\left[b-\kappa(\lambda+\nu)\right]-6m^2\delta_1^2\times\left[b-\kappa(\lambda+\nu)\right]\delta_2^2-\frac{8}{3}\Delta_0m\delta_0\delta_2s-\frac{1}{3}\Delta_0s\delta_1^2\\
-\frac{2}{3}\Delta_0m\delta_1^2s-\Delta_0m\delta_2^2h-2\Delta_0h\delta_2^2+2\alpha h\delta_2^2+\frac{1}{3}\alpha s\delta_1^2=0,\\
-\Delta_0m\delta_0\delta_1s-2\Delta_0m\delta_1\delta_2h-12m^2\delta_0\delta_1\left[b-\kappa(\lambda+\nu)\right]\delta_2^2-2\Delta_0h\delta_1\delta_2\\
+2\alpha h\delta_1\delta_2-4m^2\delta_1^3\left[b-\kappa(\lambda+\nu)\right]\delta_2=0,
\end{gathered} \quad (91)$$

$$-\frac{1}{2}\Delta_0 h\delta_1^2 + m^2\delta_2^2\Delta_2 - m^2\delta_1^4[b-\kappa(\lambda+\nu)]$$
$$-3\Delta_0 m\delta_0\delta_2 h + \frac{1}{2}\alpha h\delta_1^2 - 12m^2\delta_0\delta_1^2[b-\kappa(\lambda+\nu)]\delta_2$$
$$-\frac{1}{2}\Delta_0 m\delta_1^2 h - 4\Delta_0 p\delta_2^2$$
$$+4\alpha p\delta_2^2 - 6m^2\delta_0^2[b-\kappa(\lambda+\nu)]\delta_2^2 = 0$$
$$4\alpha p\delta_1\delta_2 - \Delta_0 m\delta_0\delta_1 h - \Delta_0 m\delta_1\delta_2 p$$
$$+2m^2\delta_1\delta_2\Delta_2 - 4m^2\delta_0\delta_1^3[b-\kappa(\lambda+\nu)]$$
$$-12m^2\delta_0^2[b-\kappa(\lambda+\nu)]\delta_1\delta_2 - 4\Delta_0 p\delta_1\delta_2 = 0,$$
$$m^2\delta_1^2\Delta_2 - 4\Delta_0 r\delta_2^2 + 4\alpha r\delta_2^2 + \alpha p\delta_1^2$$
$$+2\Delta_0 m\delta_2^2 r - \Delta_0 p\delta_1^2 - 4\Delta_0 m\delta_0\delta_2 p$$
$$-6m^2\delta_0^2[b-\kappa(\lambda+\nu)]\delta_1^2 + 2m^2\delta_0\delta_2\Delta_2$$
$$-4m^2\delta_0^3[b-\kappa(\lambda+\nu)]\delta_2 = 0,$$
$$-\Delta_0 m\delta_0\delta_1 p - 4\Delta_0 r\delta_1\delta_2 + 4\alpha r\delta_1\delta_2$$
$$-4m^2\delta_0^3[b-\kappa(\lambda+\nu)]\delta_1$$
$$+2m^2\delta_0\delta_1\Delta_2 + 2\Delta_0 m\delta_1\delta_2 r = 0,$$
$$-2\Delta_0 m\delta_0\delta_2 r + \Delta_0 m r\delta_1^2 - m^2\delta_0^4[b-\kappa(\lambda+\nu)]$$
$$-\Delta_0 r\delta_1^2 + \alpha r\delta_1^2 + m^2\delta_0^2\Delta_2 = 0.$$

With the aid of the solutions (8)–(29), we have the following types of solutions:

Type 1. Substituting $s = \frac{3h^2}{16p}$, $r = \frac{16p^2}{27h}$, into the algebraic equations (91) and solve them by Maple, one gets the following results:

$$\delta_0 = \epsilon\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}},$$
$$\delta_1 = 0,\quad \delta_2 = \frac{\epsilon h\Delta_0}{2m\Delta_2}\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}}, \tag{92}$$
$$p = \frac{3m\Delta_2}{2\Delta_0},\quad h = h,\quad \alpha = \left(1+\frac{1}{2}m\right)\Delta_0$$

provided $\Delta_2[b-\kappa(\lambda+\nu)] < 0$ and $\epsilon = \pm 1$.

If we substitute (92) along with (8)–(12) into Eq. (90), then Eq. (86) has the following solutions:

6.1.1. Soliton solutions.

$$q(x,t) = \left\{\epsilon\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}}\right.$$
$$\left.\times\left[1-\frac{4\tanh^2\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}{3+\tanh^2\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{93}$$

and

$$q(x,t) = \left\{\epsilon\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}}\right.$$
$$\left.\times\left[1-\frac{4\coth^2\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}{3+\coth^2\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{94}$$

provided $\Delta_2[b-\kappa(\lambda+\nu)] < 0, \Delta_0\Delta_2 < 0$ and $\epsilon = \pm 1$.

6.1.2. Periodic solutions.

$$q(x,t) = \left\{\epsilon\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}}\right.$$
$$\left.\times\left[1+\frac{4\tan^2\left(\epsilon\sqrt{\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}{3-\tan^2\left(\epsilon\sqrt{\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{95}$$

or

$$q(x,t) = \left\{\epsilon\sqrt{-\frac{\Delta_2}{3[b-\kappa(\lambda+\nu)]}}\right.$$
$$\left.\times\left[1+\frac{4\cot^2\left(\epsilon\sqrt{\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}{3-\cot^2\left(\epsilon\sqrt{\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{96}$$

provided $\Delta_2[b-\kappa(\lambda+\nu)] < 0$, $\Delta_0\Delta_2 > 0$ and $\epsilon = \pm 1$.

Type 2. Substituting $s = \frac{3h^2}{16p}$, $r = 0$, into the algebraic equations (92) and solve them by Maple, we get the following results:

$$\delta_0 = \epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}},$$
$$\delta_1 = 0,\quad \delta_2 = -\frac{\epsilon h\Delta_0}{2m\Delta_2}\sqrt{\frac{\Delta_2}{[b-\kappa(\lambda+\nu)]}}, \tag{97}$$
$$p = -\frac{m\Delta_2}{2\Delta_0},\quad h = h,\quad \alpha = \left(1+\frac{1}{2}m\right)\Delta_0,$$

provided $\Delta_2[b-\kappa(\lambda+\nu)] > 0$ and $\epsilon = \pm 1$.

If we substitute (97) along with (14) and (15) into Eq. (90), then Eq. (86) has following solutions.

6.1.3. Dark and singular solitons.

$$q(x,t)=\left\{\frac{\epsilon}{2}\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1+\tanh\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{98}$$

and

$$q(x,t)=\left\{\frac{\epsilon}{2}\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\times\left[1+\coth\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{99}$$

respectively, provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

Type 3. Substituting $r=0$, into the algebraic Eqs. (92) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}},\quad \delta_1=0,\\ \delta_2=-\frac{3\epsilon h\Delta_0}{4m\Delta_2}\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}},\quad p=-\frac{m\Delta_2}{2\Delta_0},\\ s=-\frac{9h^2\Delta_0}{32m\Delta_2},\quad h=h,\quad \alpha=\left(1-\frac{1}{2}m\right)\Delta_0,\end{gathered}\tag{100}$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$ and $\epsilon=\pm1$.

If we substitute (100) along with (16)–(29) into Eq. (90), then Eq. (86) has the following solutions.

6.1.4. Soliton solution.

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\left[1-\frac{12\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}{16-3\left[1+\tanh\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{101}$$

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\left[1+\frac{12\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}{16-3\left[1+\coth\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{102}$$

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\left[1-\frac{3\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}{4+2\sqrt{3}\tanh\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{103}$$

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\left[1+\frac{3\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}{4+2\sqrt{3}\coth\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{104}$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

6.1.5. Bright soliton.

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\left[1+\frac{3}{\cosh\left(\epsilon\sqrt{-\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)-2}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{105}$$

and

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\left[1+\frac{3}{2\cosh^2\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)-3}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{106}$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]>0, \Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

6.1.6. Singular soliton.

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\left[1-\frac{3}{2\sinh^2\left(\epsilon\sqrt{-\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)+3}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{107}$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

6.1.7. Periodic solutions.

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\left[1+\frac{3\sec^2\left(\epsilon\sqrt{\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}{2-3\sec^2\left(\epsilon\sqrt{\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{108}$$

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\left[1+\frac{3\mathrm{cosec}^2\left(\epsilon\sqrt{\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}{2-3\mathrm{cosec}^2\left(\epsilon\sqrt{\frac{m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{109}$$

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\left[1+\frac{3\sec\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}{1-2\sec\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{110}$$

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\Delta_2}{b-\kappa(\lambda+\nu)}}\left[1+\frac{3\mathrm{cosec}\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}{1-2\mathrm{cosec}\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{111}$$

provided $\Delta_2[b-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2>0$ and $\epsilon=\pm1$.

6.2. Addendum to Kudryashov's Method

According to this method, we balance UU'' with U^4 in Eq. (89), one gets the relation:

$$2N+2p=4N\Rightarrow N=p. \tag{112}$$

Now, we will discuss the following cases:

Case 1. If we choose $p=1$, then $N=1$. Thus, we deduce that from (31) that Eq. (89) has the formal solution:

$$U(\xi)=\beta_0+\beta_1R(\xi), \tag{113}$$

where β_0 and β_1 are constants to be determined, such that $\beta_1\neq0$ and the function $R(\xi)$ satisfies the auxiliary ODE (32). Substituting (113) along with (32) into Eq. (89), collecting all the coefficients of each power of $[R(\xi)]^{m_1}[R'(\xi)]^j$, $(m_1=0,1,2,\ldots,4,\ j=0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
\Delta_0\beta_1^2m\chi\ln^2k-\beta_1^2\alpha\chi\ln^2k\\
+\Delta_0\beta_1^2\chi\ln^2k-m^2\beta_1^4[b-\kappa(\lambda+\nu)]=0,\\
-4m^2\beta_0\beta_1^3[b-\kappa(\lambda+\nu)]+2\Delta\beta_1m\beta_0\chi\ln^2k=0,\\
-6m^2\beta_0^2[b-\kappa(\lambda+\nu)]\beta_1^2+\beta_1^2\alpha\ln^2k\\
+m^2\beta_1^2\Delta_2-\Delta_0\beta_1^2\ln^2k=0,\\
4m^2\beta_0^3[b-\kappa(\lambda+\nu)]\beta_1\\
-\Delta_0\beta_1m\beta_0\ln^2k+2m^2\beta_0\beta_1\Delta_2=0,\\
-m^2\beta_0^4[b-\kappa(\lambda+\nu)]+m^2\beta_0^2\Delta_2=0.
\end{gathered} \tag{114}$$

On solving the above algebraic Eqs. (114) by using the Maple, one gets the following results:

$$\begin{gathered}
\beta_0=0,\quad \beta_1=\epsilon\sqrt{\frac{\left(m\Delta_2+\Delta_0\ln^2k\right)\chi}{m[b-\kappa(\lambda+\nu)]}},\\
\alpha=\frac{-m^2\Delta_2+\Delta_0\ln^2k}{\ln^2k},
\end{gathered} \tag{115}$$

provided $\left(m\Delta_2+\Delta_0\ln^2k\right)[b-\kappa(\lambda+\nu)]\chi>0$ and $\epsilon=\pm1$.

Substituting (115) along with (33) into Eq. (113), one gets the solutions of Eq. (86) in the form:

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\left(m\Delta_2+\Delta_0\ln^2 k\right)\chi}{m[b-\kappa(\lambda+\nu)]}}\left[\frac{4A}{4A^2\exp_k[(x-ct)]+\chi\exp_k[-(x-ct)]}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{116}$$

In particular, if we set $\chi=4A^2$ in (116), then we have the bright soliton solution of Eq. (86) as:

$$q(x,t)=\left\{\epsilon\sqrt{\frac{m\Delta_2+\Delta_0\ln^2 k}{m[b-\kappa(\lambda+\nu)]}}\operatorname{sech}[(x-ct)\ln k]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{117}$$

while, if $\chi=-4A^2$, one gets the singular soliton solution of Eq. (86) as:

$$q(x,t)=\left\{\epsilon\sqrt{-\frac{m\Delta_2+\Delta_0\ln^2 k}{m[b-\kappa(\lambda+\nu)]}}\operatorname{cosech}[(x-ct)\ln k]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{118}$$

Case 2. If we choose $p=2$, then $N=2$. Thus, we deduce that from (31) that Eq. (89) has the formal solution:

$$\phi(\xi)=\beta_0+\beta_1 R(\xi)+\beta_2 R^2(\xi), \tag{119}$$

where β_0,β_1 and β_2 are constants to be determined, such that $\beta_2\neq 0$ and the function $R(\xi)$ satisfies the auxiliary ODE (32). Substituting (119) along with Eq. (32) into Eq. (89), collecting all the coefficients of each power of $[R(\xi)]^{m_2}\,[R'(\xi)]^j$, $(m_2=0,1,2,\ldots 8,\quad j=0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
4\Delta_0\chi\beta_2^2\ln^2 k-4\alpha\chi\beta_2^2\ln^2 k-m^2\beta_2^4[b-\kappa(\lambda+\nu)]+4\Delta_0 m\beta_2^2\chi\ln^2 k=0,\\
7\Delta_0\beta_1 m\beta_2\chi\ln^2 k-4\alpha\chi\beta_1\beta_2\ln^2 k-4m^2\beta_1\beta_2^3[b-\kappa(\lambda+\nu)]+4\Delta_0\chi\beta_1\beta_2\ln^2 k=0,\\
\Delta_0\beta_1^2\chi\ln^2 k+8\Delta_0 m\beta_0\beta_2\chi\ln^2 k+2\Delta_0\beta_1^2 m\chi\ln^2 k-6m^2\beta_1^2[b-\kappa(\lambda+\nu)]\beta_2^2\\
-\beta_1^2\alpha\chi\ln^2 k-4m^2\beta_0\beta_2^3[b-\kappa(\lambda+\nu)]=0,\\
-4m^2\beta_1^3[b-\kappa(\lambda+\nu)]\beta_2+3\Delta_0\beta_1 m\beta_0\chi\ln^2 k-12m^2\beta_0\beta_1[b-\kappa(\lambda+\nu)]\beta_2^2=0,\\
-12m^2\beta_0\beta_1^2[b-\kappa(\lambda+\nu)]\beta_2+m^2\beta_2^2\Delta_2-6m^2\beta_0^2[b-\kappa(\lambda+\nu)]\beta_2^2\\
-m^2\beta_1^4[b-\kappa(\lambda+\nu)]+4\alpha\beta_2^2\ln^2 k-4\Delta_0\beta_2^2=0,\\
-4\Delta_0\beta_1\beta_2\ln^2 k-\Delta_0\beta_1 m\beta_2\ln^2 k+4\alpha\beta_1\beta_2\ln^2 k+2m^2\beta_1\beta_2\Delta_2-12m^2\beta_0^2[b-\kappa(\lambda+\nu)]\beta_1\beta_2\\
-4m^2\beta_0\beta_1^3[b-\kappa(\lambda+\nu)]=0,\\
m^2\beta_1^2\Delta_2-6m^2\beta_0^2[b-\kappa(\lambda+\nu)]\beta_1^2+2m^2\beta_0\beta_2\Delta_2-4\Delta_0 m\beta_0\beta_2\ln^2 k+\beta_1^2\alpha\ln^2 k\\
-4m^2\beta_0^3[b-\kappa(\lambda+\nu)]\beta_2-\Delta_0\beta_1^2\ln^2 k=0,\\
-4m^2\beta_0^3[b-\kappa(\lambda+\nu)]\beta_1-\Delta_0\beta_1 m\beta_0\ln^2 k+2m^2\beta_0\beta_1\Delta_2=0,\\
-m^2\beta_0^4[b-\kappa(\lambda+\nu)]+m^2\beta_0^2\Delta_2=0.
\end{gathered} \tag{120}$$

On solving the above algebraic Eqs. (120) by using the Maple, one gets the following results:

$$\beta_0=0,\quad \beta_1=0,\quad \beta_2=\epsilon\sqrt{\frac{\left(m\Delta_2+4\Delta_0\ln^2 k\right)\chi}{m[b-\kappa(\lambda+\nu)]}},\quad \alpha=\frac{-m^2\Delta_2+4\Delta_0\ln^2 k}{4\ln^2 k}, \tag{121}$$

provided $\left(m\Delta_2+4\Delta_0\ln^2 k\right)[b-\kappa(\lambda+\nu)]\chi>0$ and $\epsilon=\pm 1$.

Substituting (121) along with (33) into Eq. (119), one gets the solutions of Eq. (86) in the form:

$$q(x,t)=\left\{\epsilon\sqrt{\frac{\left(m\Delta_2+4\Delta_0\ln^2 k\right)\chi}{m[b-\kappa(\lambda+\nu)]}}\left[\frac{4A}{4A^2\exp_k[2(x-ct)]+\chi\exp_k[-2(x-ct)]}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{122}$$

In particular, if we set $\chi = 4A^2$ in (122), then we have the bright soliton solution of Eq. (86) as:

$$q(x,t) = \left\{\epsilon\sqrt{\frac{m\Delta_2 + 4\Delta_0 \ln^2 k}{m[b - \kappa(\lambda + \nu)]}} \operatorname{sech}[2(x - ct)\ln k]\right\}^{\frac{1}{m}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{123}$$

while, if $\chi = -4A^2$, one gets the singular soliton solution of Eq. (86) as:

$$q(x,t) = \left\{\epsilon\sqrt{-\frac{m\Delta_2 + 4\Delta_0 \ln^2 k}{m[b - \kappa(\lambda + \nu)]}} \operatorname{cosech}[2(x - ct)\ln k]\right\}^{\frac{1}{m}} e^{i(-\kappa x + \omega t + \theta_0)}. \tag{124}$$

Similarly, we can find many other solutions by choosing other values for p and N.

7. PARABOLIC LAW

For the parabolic law nonlinearity, we have

$$F(\phi) = b_1\phi + b_2\phi^2, \tag{125}$$

where b_1 and b_2 are constants such that $b_2 \neq 0$. Parabolic law nonlinearity arises in the context of plasma physics to study the nonlinear interaction between Langmuir waves and electrons and describes the nonlinear interaction between the high frequency Langmuir waves and the ion-acoustic waves by pondermotive forces.

Equation (1) corresponding to parabolic law nonlinearity (125) is given by:

$$\begin{aligned} iq_t + iaq_{xxx} + \left(b_1|q|^2 + b_2|q|^4\right)q = \alpha\frac{|q_x|^2}{q^*} \\ + \frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \\ + i\left[\delta q_x + \lambda\left(|q|^{2m}q\right)_x + \mu\left(|q|^{2m}\right)_x q + \nu|q|^{2m}q_x\right], \end{aligned} \tag{126}$$

where Eq. (41) reduces to:

$$\begin{aligned} \Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2 \\ + b_1\phi^4 - \kappa(\lambda + \nu)\phi^{2m+2} + b_2\phi^6 = 0. \end{aligned} \tag{127}$$

For integrability, one must select $m = 1$. This leads to the modification of Eq. (1) corresponding to parabolic law nonlinearity as:

$$\begin{aligned} iq_t + iaq_{xxx} + \left(b_1|q|^2 + b_2|q|^4\right)q = \alpha\frac{|q_x|^2}{q^*} \\ + \frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \\ + i\left[\delta q_x + \lambda\left(|q|^2q\right)_x + \mu\left(|q|^2\right)_x q + \nu|q|^2q_x\right], \end{aligned} \tag{128}$$

Consequently, Eq. (127) changes to:

$$\begin{aligned} \Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2 \\ + [b_1 - \kappa(\lambda + \nu)]\phi^4 + b_2\phi^6 = 0. \end{aligned} \tag{129}$$

Balancing $\phi\phi''$ and ϕ^6 in Eq. (129), gives the balance number $N = \frac{1}{2}$. Since the balance number is not integer, then we take into consideration the transformation

$$\phi(\xi) = [U(\xi)]^{\frac{1}{2}}, \tag{130}$$

where $U(\xi)$ is a new positive function of ξ. Substituting (130) into (129), we have the new equation

$$\begin{aligned} 2\Delta_0UU'' - (\alpha + \Delta_0)U'^2 - 4\Delta_2U^2 \\ + 4[b_1 - \kappa(\lambda + \nu)]U^3 + 4b_2U^4 = 0. \end{aligned} \tag{131}$$

In the next two subsections, we will solve Eq. (131) using the following two methods.

7.1. New Mapping Method

According to the new mapping method, we balance UU'' with U^4 in Eq. (131) yields the balance number $N = 1$. Now, from (6), the solution of Eq. (131) has the same formal solution (90). Substituting (90) along with (7) into Eq. (131), collecting all the coefficients of $F^l(\xi)[F'(\xi)]^j$ ($l = 0,1,\ldots,8$, $j = 0,1$) and setting them to zero, we have the following algebraic equations:

$$-3\Delta_0 s\delta_2^2 + \alpha s\delta_2^2 - 3b_2\delta_2^4 = 0,$$

$$-6\Delta_0 s\delta_1\delta_2 + \frac{4}{3}\alpha s\delta_1\delta_2 - 16b_2\delta_1\delta_2^3 = 0,$$

$$\begin{aligned} -16b_2\delta_0\delta_2^3 + \frac{1}{3}\alpha s\delta_1^2 - \frac{5}{3}\Delta_0 s\delta_1^2 \\ - 4[b_1 - \kappa(\lambda + \nu)]\delta_2^3 + 2\alpha h\delta_2^2 \\ - \frac{16}{3}\Delta_0\delta_0\delta_2 s - 4\Delta_0 h\delta_2^2 - 24b_2\delta_1^2\delta_2^2 = 0, \end{aligned}$$

$$\begin{aligned} -16b_2\delta_1^3\delta_2 - 12[b_1 - \kappa(\lambda + \nu)]\delta_1\delta_2^2 \\ - 48b_2\delta_0\delta_1\delta_2^2 - 2\Delta_0\delta_0\delta_1 s \\ + 2\alpha h\delta_1\delta_2 - 6\Delta_0 h\delta_1\delta_2 = 0, \end{aligned}$$

$$\begin{aligned} -48b_2\delta_0\delta_1^2\delta_2 - 12[b_1 - \kappa(\lambda + \nu)]\delta_1^2\delta_2 \\ + 4\Delta_2\delta_2^2 - \frac{3}{2}\Delta_0 h\delta_1^2 - 6\Delta_0\delta_0\delta_2 h - 4b_2\delta_1^4 \end{aligned}$$

$$
\begin{gathered}
-12[b_1-\kappa(\lambda+\nu)]\delta_0\delta_2^2-4\Delta_0 p\delta_2^2 \\
+\frac{1}{2}\alpha h\delta_1^2-24b_2\delta_0^2\delta_2^2+4\alpha p\delta_2^2=0, \\
-16b_2\delta_0\delta_1^3+4\alpha p\delta_1\delta_2-48b_2\delta_0^2\delta_1\delta_2 \\
-4[b_1-\kappa(\lambda+\nu)]\delta_1^3+8\Delta_2\delta_1\delta_2-6\Delta_0 p\delta_1\delta_2 \\
-24[b_1-\kappa(\lambda+\nu)]\delta_0\delta_1\delta_2-2\Delta_0\delta_0\delta_1 h=0, \\
4\alpha r\delta_2^2+\alpha p\delta_1^2+4\Delta_2\delta_1^2-\Delta_0 p\delta_1^2 \\
+8\Delta_2\delta_0\delta_2-12[b_1-\kappa(\lambda+\nu)]\delta_0^2\delta_2-16b_2\delta_0^3\delta_2 \\
-12[b_1-\kappa(\lambda+\nu)]\delta_0\delta_1^2-24b_2\delta_0^2\delta_1^2-8\Delta_0\delta_0\delta_2 p=0, \\
4\alpha r\delta_1\delta_2+8\Delta_2\delta_0\delta_1-2\Delta_0\delta_0\delta_1 p \\
-12[b_1-\kappa(\lambda+\nu)]\delta_0^2\delta_1-16b_2\delta_0^3\delta_1=0, \\
-4\Delta_0\delta_0\delta_2 r-4[b_1-\kappa(\lambda+\nu)]\delta_0^3 \\
-4b_2\delta_0^4+4\Delta_2\delta_0^2+\Delta_0 r\delta_1^2+\alpha r\delta_1^2=0.
\end{gathered} \tag{132}
$$

With the aid of the solutions (8)–(29), we have the following types of solutions:

Type 1. Substituting $s=\frac{3h^2}{16p}$, $r=\frac{16p^2}{27h}$, into the algebraic Eqs. (132) and solve them by Maple, one gets the following results:

$$
\begin{gathered}
\delta_0=-\frac{2\Delta_2}{3[b_1-\kappa(\lambda+\nu)]},\quad \delta_1=0, \\
\delta_2=-\frac{h\Delta_0}{12[b_1-\kappa(\lambda+\nu)]},\quad p=\frac{6\Delta_2}{\Delta_0},\quad h=h, \\
b_2=-\frac{9[b_1-\kappa(\lambda+\nu)]^2}{4\Delta_2},\quad \alpha=\frac{3}{2}\Delta_0,
\end{gathered} \tag{133}
$$

provided $b_1-\kappa(\lambda+\nu)\neq 0$ and $\Delta_2\neq 0$.

If we substitute (133) along with (8)–(12) into Eq. (90), then Eq. (128) has the following solutions.

7.1.1. Soliton solutions.

$$
q(x,t)=\left\{-\frac{2\Delta_2}{3[b_1-\kappa(\lambda+\nu)]}\left[1-\frac{4\tanh^2\left(\epsilon\sqrt{-\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}{3+\tanh^2\left(\epsilon\sqrt{-\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{134}
$$

and

$$
q(x,t)=\left\{-\frac{2\Delta_2}{3[b_1-\kappa(\lambda+\nu)]}\left[1-\frac{4\coth^2\left(\epsilon\sqrt{-\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}{3+\coth^2\left(\epsilon\sqrt{-\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{135}
$$

provided $\Delta_2[b_1-\kappa(\lambda+\nu)]<0, \Delta_0\Delta_2<0$ and $\epsilon=\pm 1$.

7.1.2. Periodic solutions.

$$
q(x,t)=\left\{-\frac{2\Delta_2}{3[b_1-\kappa(\lambda+\nu)]}\left[1+\frac{4\tan^2\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}{3-\tan^2\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{136}
$$

and

$$
q(x,t)=\left\{-\frac{2\Delta_2}{3[b_1-\kappa(\lambda+\nu)]}\left[1+\frac{4\cot^2\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}{3-\cot^2\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{137}
$$

provided $\Delta_2[b_1-\kappa(\lambda+\nu)]<0, \Delta_0\Delta_2>0$ and $\epsilon=\pm 1$.

Type 2. Substituting $s = \frac{3h^2}{16p}$, $r = 0$, into the algebraic Eqs. (132) and solve them by Maple, we get the following results:

$$\delta_0 = -\frac{2\Delta_2}{b_1 - \kappa(\lambda+\nu)}, \quad \delta_1 = 0,$$
$$\delta_2 = \frac{h\Delta_0}{4[b_1 - \kappa(\lambda+\nu)]}, \quad p = -\frac{2\Delta_2}{\Delta_0}, \quad h = h, \tag{138}$$
$$b_2 = \frac{3[b_1 - \kappa(\lambda+\nu)]^2}{4\Delta_2}, \quad \alpha = \frac{3}{2}\Delta_0,$$

provided $b_1 - \kappa(\lambda+\nu) \neq 0, \Delta_0 \neq 0$ and $\Delta_2 \neq 0$.

If we substitute (138) along with (14) and (15) into Eq. (90), then Eq. (128) has the following solutions.

7.1.3. Dark and singular solitons.

$$q(x,t) = \left\{-\frac{\Delta_2}{b_1 - \kappa(\lambda+\nu)} \times \left[1 + \tanh\left(\epsilon\sqrt{-\frac{2\Delta_2}{\Delta_0}}(x - ct)\right)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{139}$$

and

$$q(x,t) = \left\{-\frac{\Delta_2}{b_1 - \kappa(\lambda+\nu)} \times \left[1 + \coth\left(\varepsilon\sqrt{-\frac{2\Delta_2}{\Delta_0}}(x - ct)\right)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{140}$$

respectively, provided $\Delta_2[b_1 - \kappa(\lambda+\nu)] < 0, \Delta_0\Delta_2 < 0$ and $\epsilon = \pm 1$.

Type 3. Substituting $r = 0$, into the algebraic Eqs. (132) and solve them by Maple, we get the following results:

$$\delta_0 = \frac{6\Delta_2}{b_1 - \kappa(\lambda+\nu)}, \quad \delta_1 = 0,$$
$$\delta_2 = \frac{3h\Delta_0}{4[b_1 - \kappa(\lambda+\nu)]}, \quad p = \frac{2\Delta_2}{\Delta_0}, \quad s = \frac{3h^2\Delta_0}{32\Delta_2}, \tag{141}$$
$$h = h, \quad b_2 = -\frac{5[b_1 - \kappa(\lambda+\nu)]^2}{36\Delta_2}, \quad \alpha = \frac{1}{2}\Delta_0,$$

provided $b_1 - \kappa(\lambda+\nu) \neq 0, \Delta_0 \neq 0$ and $\Delta_2 \neq 0$.

If we substitute (141) along with (16)–(29) into Eq. (90), then Eq. (128) has the following solutions.

7.1.4. Soliton solutions.

$$q(x,t) = \left\{\frac{6\Delta_2}{b_1 - \kappa(\lambda+\nu)}\left[1 - \frac{2\operatorname{sech}^2\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x - ct)\right)}{4 - \left[1 + \tanh\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x - ct)\right)\right]^2}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{142}$$

$$q(x,t) = \left\{\frac{6\Delta_2}{b_1 - \kappa(\lambda+\nu)}\left[1 + \frac{2\operatorname{cosech}^2\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x - ct)\right)}{4 - \left[1 + \coth\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x - ct)\right)\right]^2}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{143}$$

$$q(x,t) = \left\{\frac{3\Delta_2}{b_1 - \kappa(\lambda+\nu)}\left[2 - \frac{\operatorname{sech}^2\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x - ct)\right)}{1 + \tanh\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x - ct)\right)}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{144}$$

$$q(x,t) = \left\{\frac{3\Delta_2}{b_1 - \kappa(\lambda+\nu)}\left[2 + \frac{\operatorname{cosech}^2\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x - ct)\right)}{1 + \coth\left(\epsilon\sqrt{\frac{2\Delta_2}{\Delta_0}}(x - ct)\right)}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{145}$$

provided $\Delta_2[b_1 - \kappa(\lambda+\nu)] > 0, \Delta_0\Delta_2 > 0$ and $\epsilon = \pm 1$.

7.2. Addendum to Kudryashov's Method

According to this method, we balance UU'' with U^4 in Eq. (131), one gets the relation:

$$2N + 2p = 4N \Rightarrow N = p. \quad (146)$$

Now, we will discuss the following cases:

Case 1. If we choose $p = 1$, then $N = 1$. Thus, we deduce that from (31) that Eq. (131) has the same formal solution (113). Substituting (113) along with (32) into Eq. (131), collecting all the coefficients of each power of $[R(\xi)]^{m_1}\ [R'(\xi)]^j$, $(m_1 = 0,1,2,\ldots,4,\ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
-\alpha\beta_1^2\chi\ln^2 k + 3\Delta_0\beta_1^2\chi\ln^2 k - 4b_2\beta_1^4 = 0,\\
-4[b_1 - \kappa(\lambda+\nu)]\beta_1^3 + 4\Delta_0\beta_1\beta_0\chi\ln^2 k - 16b_2\beta_0\beta_1^3 = 0,\\
\alpha\beta_1^2\ln^2 k - 12[b_1 - \kappa(\lambda+\nu)]\beta_0\beta_1^2\\
+ 4\Delta_2\beta_1^2 - \Delta_0\beta_1^2\ln^2 k - 24b_2\beta_0^2\beta_1^2 = 0,\\
8\Delta_2\beta_0\beta_1 - 16b_2\beta_0^3\beta_1 - 2\Delta_0\beta_1\beta_0\ln^2 k\\
- 12[b_1 - \kappa(\lambda+\nu)]\beta_0^2\beta_1 = 0,\\
-4b_2\beta_0^4 - 4[b_1 - \kappa(\lambda+\nu)]\beta_0^3 + 4\Delta_2\beta_0^2 = 0.
\end{gathered} \quad (147)$$

On solving the above algebraic Eqs. (147) by using the Maple, one gets the following results:

$$\begin{gathered}
\beta_0 = -\frac{3\Delta_0\ln^2 k}{2[b_1 - \kappa(\lambda+\nu)]},\quad \beta_1 = \frac{3\epsilon\Delta_0\sqrt{2\chi}\ln^2 k}{2[b_1 - \kappa(\lambda+\nu)]},\quad \alpha = -\Delta_0,\\
\Delta_2 = -\Delta_0\ln^2 k,\quad b_2 = \frac{2[b_1 - \kappa(\lambda+\nu)]^2}{9\Delta_0\ln^2 k},
\end{gathered} \quad (148)$$

provided $\chi > 0, [b_1 - \kappa(\lambda+\nu)] \neq 0, \Delta_0 \neq 0$ and $\epsilon = \pm 1$.

Substituting (148) along with (33) into Eq. (113), one gets the solutions of Eq. (128) in the form:

$$q(x,t) = \left\{-\frac{3\Delta_0\ln^2 k}{2[b_1 - \kappa(\lambda+\nu)]}\left[1 + \frac{4\epsilon A\sqrt{2\chi}}{4A^2\exp_k[(x-ct)] + \chi\exp_k[-(x-ct)]}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \quad (149)$$

provided $\Delta_0[b_1 - \kappa(\lambda+\nu)] < 0, \chi < 0$ and $\epsilon = \pm 1$.

In particular, if we set $\chi = 4A^2$ in (149), then we have the bright soliton solution of Eq. (128) as:

$$q(x,t) = \left\{-\frac{3\Delta_0\ln^2 k}{b_1 - \kappa(\lambda+\nu)}\left[1 + \epsilon\sqrt{2}\operatorname{sech}[(x-ct)\ln k]\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}. \quad (150)$$

Case 2. If we choose $p = 2$, then $N = 2$. Thus, we deduce that from (31) that Eq. (131) has the same formal solution (119). Substituting (119) along with Eq. (32) into Eq. (131), collecting all the coefficients of each power of $[R(\xi)]^{m_2}\ [R'(\xi)]^j$, $(m_2 = 0,1,2,\ldots,8,\ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
-4\alpha\chi\beta_2^2\ln^2 k + 12\Delta_0\chi\beta_2^2\ln^2 k - 4b_2\beta_2^4 = 0,\\
-16b_2\beta_1\beta_2^3 - 4\alpha\chi\beta_1\beta_2\ln^2 k\\
+ 18\Delta_0\chi\beta_1\beta_2\ln^2 k = 0,\\
-24b_2\beta_1^2\beta_2^2 + 16\Delta_0\beta_0\beta_2\chi\ln^2 k\\
+ 5\Delta_0\beta_1^2\chi\ln^2 k - 4[b_1 - \kappa(\lambda+\nu)]\beta_2^3\\
- 16b_2\beta_0\beta_2^3 - \alpha\beta_1^2\chi\ln^2 k = 0,\\
-48b_2\beta_0\beta_1\beta_2^2 - 16b_2\beta_1^3\beta_2 - 12[b_1 - \kappa(\lambda+\nu)]\beta_1\beta_2^2\\
+ 6\Delta_0\beta_1\beta_0\chi\ln^2 k = 0,\\
-48b_2\beta_0\beta_1^2\beta_2 + 4\alpha\beta_2^2\ln^2 k - 4\Delta_0\beta_2^2\ln^2 k\\
- 12[b_1 - \kappa(\lambda+\nu)]\beta_0\beta_2^2 - 24b_2\beta_0^2\beta_2^2\\
-12[b_1 - \kappa(\lambda+\nu)]\beta_1^2\beta_2 + 4\Delta_2\beta_2^2 - 4b_2\beta_1^4 = 0,\\
4\alpha\beta_1\beta_2\ln^2 k - 24[b_1 - \kappa(\lambda+\nu)]\beta_0\beta_1\beta_2\\
+ 8\Delta_2\beta_1\beta_2 - 48b_2\beta_0^2\beta_1\beta_2 - 6\Delta_0\beta_1\beta_2\ln^2 k\\
-4[b_1 - \kappa(\lambda+\nu)]\beta_1^3 - 16b_2\beta_0\beta_1^3 = 0,\\
-16b_2\beta_0^3\beta_2 - 12[b_1 - \kappa(\lambda+\nu)]\beta_0^2\beta_2\\
- 12[b_1 - \kappa(\lambda+\nu)]\beta_0\beta_1^2 - \Delta_0\beta_1^2\ln^2 k - 24b_2\beta_0^2\beta_1^2\\
+ 4\Delta_2\beta_1^2 + 8\Delta_2\beta_0\beta_2 - 8\Delta_0\beta_0\beta_2\ln^2 k + \alpha\beta_1^2\ln^2 k = 0,\\
8\Delta_2\beta_0\beta_1 - 16b_2\beta_0^3\beta_1 - 2\Delta_0\beta_1\beta_0\ln^2 k\\
- 12[b_1 - \kappa(\lambda+\nu)]\beta_0^2\beta_1 = 0,\\
-4[b_1 - \kappa(\lambda+\nu)]\beta_0^3 - 4b_2\beta_0^4 + 4\Delta_2\beta_0^2 = 0.
\end{gathered} \quad (151)$$

On solving the above algebraic Eqs. (151) by using the Maple, one gets the following results:

$$\beta_0 = -\frac{6\Delta_0 \ln^2 k}{b_1 - \kappa(\lambda+\nu)},\quad \beta_1 = 0,\quad \beta_2 = \frac{6\epsilon\Delta_0\sqrt{2\chi}\ln^2 k}{b_1 - \kappa(\lambda+\nu)},$$
$$\alpha = -\Delta_0,\quad \Delta_2 = -4\Delta_0 \ln^2 k,\quad b_2 = \frac{[b_1 - \kappa(\lambda+\nu)]^2}{18\Delta_0 \ln^2 k}, \tag{152}$$

provided $\chi > 0, [b_1 - \kappa(\lambda+\nu)] \neq 0, \Delta_0 \neq 0$ and $\epsilon = \pm 1$.

Substituting (152) along with (33) into Eq. (119), one gets the solutions of Eq. (128) in the form:

$$q(x,t) = \left\{-\frac{6\Delta_0 \ln^2 k}{b_1 - \kappa(\lambda+\nu)}\left[1 + \frac{4\epsilon A\sqrt{2\chi}}{4A^2 \exp_k[2(x-ct)] + \chi\exp_k[-2(x-ct)]}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{153}$$

provided $\Delta_0[b_1 - \kappa(\lambda+\nu)] < 0$ and $\epsilon = \pm 1$.

In particular, if we set $\chi = 4A^2$ in (153), then we have the bright soliton solution of Eq. (128) as:

$$q(x,t) = \left\{-\frac{6\Delta_0 \ln^2 k}{b_1 - \kappa(\lambda+\nu)}\left[1 + \epsilon\sqrt{2}\operatorname{sech}[2(x-ct)\ln k]\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}. \tag{154}$$

Similarly, we can find many other solutions by choosing other values for p and N.

8. DUAL POWER LAW

For the dual power law nonlinearity, we have

$$F(\phi) = b_1\phi^n + b_2\phi^{2n}, \tag{155}$$

where b_1 and b_2 are arbitrary constants such that $b_2 \neq 0$. The dual power law nonlinearity is used to describe the soliton dynamics in photo voltaic-photo refractive materials.

Equation (1) corresponding to dual power law nonlinearity (155) is given by:

$$\begin{aligned} &iq_t + iaq_{xxx} + \left(b_1|q|^{2n} + b_2|q|^{4n}\right)q = \alpha\frac{|q_x|^2}{q^*} \\ &+ \frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \\ &+ i\left[\delta q_x + \lambda\left(|q|^{2m}q\right)_x + \mu\left(|q|^{2m}\right)_x q + \nu|q|^{2m}q_x\right], \end{aligned} \tag{156}$$

where Eq. (41) reduces to:

$$\begin{aligned} &\Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2 + b_1\phi^{2n+2} \\ &- \kappa(\lambda+\nu)\phi^{2m+2} + b_2\phi^{4n+2} = 0. \end{aligned} \tag{157}$$

For integrability, one must select $n = m$. This leads to the modification of Eq. (1) corresponding to dual power law nonlinearity as:

$$\begin{aligned} &iq_t + iaq_{xxx} + \left(b_1|q|^{2m} + b_2|q|^{4m}\right)q = \alpha\frac{|q_x|^2}{q^*} \\ &+ \frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \\ &+ i\left[\delta q_x + \lambda\left(|q|^{2m}q\right)_x + \mu\left(|q|^{2m}\right)_x q + \nu|q|^{2m}q_x\right]. \end{aligned} \tag{158}$$

Consequently, Eq. (157) changes to:

$$\begin{aligned} &\Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2 \\ &+ [b_1 - \kappa(\lambda+\nu)]\phi^{2m+2} + b_2\phi^{4m+2} = 0. \end{aligned} \tag{159}$$

Balancing $\phi\phi''$ and ϕ^{4m+2} in Eq. (159), gives the balance number $N = \frac{1}{2m}$. Since the balance number is not integer, then we take into consideration the transformation

$$\phi(\xi) = [U(\xi)]^{\frac{1}{2m}}, \tag{160}$$

where $U(\xi)$ is a new positive function of ξ. Substituting (160) into (159), we have the new equation

$$\begin{aligned} &2m\Delta_0 UU'' - [\alpha + (2m-1)\Delta_0]U'^2 \\ &- 4m^2\Delta_2 U^2 + 4m^2[b_1 - \kappa(\lambda+\nu)]U^3 \\ &+ 4m^2 b_2 U^4 = 0. \end{aligned} \tag{161}$$

In the next two subsections, we will solve Eq. (161) using the following two methods.

8.1. New Mapping Method

According to the new mapping method, we balance UU'' with U^4 in Eq. (161) yields the balance number $N = 1$. Now, from (6), the solution of Eq. (161) has the same formal solution (90). Substituting (90) along with (7) into Eq. (161), collecting all the coefficients of $F^l(\xi)[F'(\xi)]^j$ $(l = 0,1,\ldots,8,\ j = 0,1)$ and setting them to zero, we have the following algebraic equations:

$$\frac{4}{3}\Delta_0 s\delta_2^2 + \frac{8}{3}\Delta_0 m\delta_2^2 s + 4m^2\delta_2^4 b_2 - \frac{4}{3}\alpha s\delta_2^2 = 0,$$
$$\frac{14}{3}\Delta_0 m\delta_1\delta_2 s + 16m^2\delta_1\delta_2^3 b_2$$

$$+\frac{4}{3}\Delta_0 s\delta_1\delta_2-\frac{4}{3}\alpha s\delta_1\delta_2=0,$$
$$2\Delta_0 m\delta_2^2 h+\frac{1}{3}\Delta_0 s\delta_1^2+\frac{16}{3}\Delta_0 m\delta_0\delta_2 s$$
$$-\frac{1}{3}\alpha s\delta_1^2+\frac{4}{3}\Delta_0 m\delta_1^2 s$$
$$+16m^2\delta_0\delta_2^3 b_2-2\alpha h\delta_2^2+2h\Delta_0\delta_2^2$$
$$+4m^2\delta_2^3[b_1-\kappa(\lambda+\nu)]+24m^2\delta_1^2 b_2\delta_2^2=0,$$
$$2h\Delta_0\delta_1\delta_2-2\alpha h\delta_1\delta_2+16m^2\delta_1^3 b_2\delta_2$$
$$+12m^2\delta_1\delta_2^2[b_1-\kappa(\lambda+\nu)]+4\Delta_0 m\delta_1\delta_2 h$$
$$+2\Delta_0 m\delta_0\delta_1 s+48m^2\delta_0\delta_1 b_2\delta_2^2=0,$$
$$\frac{1}{2}h\Delta_0\delta_1^2+12m^2\delta_1^2\delta_2[b_1-\kappa(\lambda+\nu)]+4\Delta_0 p\delta_2^2$$
$$-\frac{1}{2}\alpha h\delta_1^2-4\alpha p\delta_2^2+4m^2\delta_1^4 b_2+\Delta_0 m\delta_1^2 h$$
$$+12m^2\delta_0\delta_2^2[b_1-\kappa(\lambda+\nu)]+6\Delta_0 m\delta_0\delta_2 h \quad (162)$$
$$-4m^2\delta_2^2\Delta_2+48m^2\delta_0\delta_1^2 b_2\delta_2+24m^2\delta_0^2 b_2\delta_2^2=0,$$
$$2\Delta_0 m\delta_1\delta_2 p+48m^2\delta_0^2 b_2\delta_1\delta_2+2\Delta_0 m\delta_0\delta_1 h$$
$$-4\alpha p\delta_1\delta_2+4m^2\delta_1^3[b_1-\kappa(\lambda+\nu)]+4\Delta_0 p\delta_1\delta_2$$
$$+16m^2\delta_0\delta_1^3 b_2-8m^2\delta_1\delta_2\Delta_2$$
$$+24m^2\delta_0\delta_1[b_1-\kappa(\lambda+\nu)]\delta_2=0,$$
$$\Delta_0 p\delta_1^2-\alpha p\delta_1^2-4\Delta_0 m\delta_2^2 r+4\Delta_0 r\delta_2^2$$
$$-8m^2\delta_0\delta_2\Delta_2-4m^2\delta_1^2\Delta_2+16m^2\delta_0^3 b_2\delta_2$$
$$+12m^2\left\{\delta_0\delta_1^2+\delta_0^2\delta_2\right\}[b_1-\kappa(\lambda+\nu)]$$
$$+24m^2\delta_0^2 b_2\delta_1^2-4\alpha r\delta_2^2+8\Delta_0 m\delta_0\delta_2 p=0,$$
$$16m^2\delta_0^3 b_2\delta_1+4\Delta_0 r\delta_1\delta_2+12m^2\delta_0^2\delta_1[b_1-\kappa(\lambda+\nu)]$$
$$-8m^2\delta_0\delta_1\Delta_2-4\alpha r\delta_1\delta_2+2\Delta_0 m\delta_0\delta_1 p-4\Delta_0 m\delta_1\delta_2 r=0,$$
$$4\Delta_0 m\delta_0\delta_2 r+4m^2\delta_0^4 b_2+4m^2\delta_0^3[b_1-\kappa(\lambda+\nu)]$$
$$-4m^2\delta_0^2\Delta_2+\Delta_0 r\delta_1^2-\alpha r\delta_1^2-2\Delta_0 mr\delta_1^2=0.$$

With the aid of the solutions (8)–(29), we have the following types of solutions:

Type 1. Substituting $s=\frac{3h^2}{16p}$, $r=\frac{16p^2}{27h}$, into the algebraic Eqs. (162) and solve them by Maple, one gets the following results:

$$\delta_0=-\frac{2\Delta_2}{3[b_1-\kappa(\lambda+\nu)]},\quad \delta_1=0,$$
$$\delta_2=-\frac{h\Delta_0}{12m[b_1-\kappa(\lambda+\nu)]},\quad p=\frac{6m\Delta_2}{\Delta_0},\quad h=h, \quad (163)$$
$$b_2=-\frac{9[b_1-\kappa(\lambda+\nu)]^2}{4\Delta_2},\quad \alpha=\left(1+\frac{1}{2}m\right)\Delta_0,$$

provided $b_1-\kappa(\lambda+\nu)\neq 0$ and $\Delta_2\neq 0$.

If we substitute (163) along with (8)–(12) into Eq. (90), then Eq. (158) has the following solutions.

8.1.1. Soliton solutions.

$$q(x,t)=\left\{-\frac{2\Delta_2}{3[b_1-\kappa(\lambda+\nu)]}\times\left[1-\frac{4\tanh^2\left(\epsilon\sqrt{-\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}{3+\tanh^2\left(\epsilon\sqrt{-\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x+\omega t+\theta_0)}, \quad (164)$$

and

$$q(x,t)=\left\{-\frac{2\Delta_2}{3[b_1-\kappa(\lambda+\nu)]}\times\left[1-\frac{4\coth^2\left(\epsilon\sqrt{-\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}{3+\coth^2\left(\epsilon\sqrt{-\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x+\omega t+\theta_0)}, \quad (165)$$

provided $\Delta_2[b_1-\kappa(\lambda+\nu)]<0, \Delta_0\Delta_2<0$ and $\epsilon=\pm 1$.

8.1.2. Periodic solutions.

$$q(x,t)=\left\{-\frac{2\Delta_2}{3[b_1-\kappa(\lambda+\nu)]}\times\left[1+\frac{4\tan^2\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}{3-\tan^2\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x+\omega t+\theta_0)}, \quad (166)$$

and

$$q(x,t)=\left\{-\frac{2\Delta_2}{3[b_1-\kappa(\lambda+\nu)]}\times\left[1+\frac{4\cot^2\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}{3-\cot^2\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x+\omega t+\theta_0)}, \quad (167)$$

provided $\Delta_2[b_1-\kappa(\lambda+\nu)]<0$, $\Delta_0\Delta_2>0$ and $\epsilon=\pm 1$.

Type 2. Substituting $s=\dfrac{3h^2}{16p}$, $r=0$, into the algebraic equations (162) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=-\frac{2\Delta_2}{b_1-\kappa(\lambda+\nu)},\quad \delta_1=0,\\ \delta_2=\frac{h\Delta_0}{4m[b_1-\kappa(\lambda+\nu)]},\quad p=-\frac{2m\Delta_2}{\Delta_0},h=h,\\ b_2=\frac{3[b_1-\kappa(\lambda+\nu)]^2}{4\Delta_2},\quad \alpha=\left(1+\frac{m}{2}\right)\Delta_0,\end{gathered}\tag{168}$$

provided $b_1-\kappa(\lambda+\nu)\neq 0,\Delta_0\neq 0$ and $\Delta_2\neq 0$.

If we substitute (168) along with (14) and (15) into Eq. (90), then Eq. (158) has the following solutions.

8.1.3. Dark and singular solitons.

$$\begin{gathered}q(x,t)=\left\{-\frac{\Delta_2}{b_1-\kappa(\lambda+\nu)}\right.\\ \left.\times\left[1+\tanh\left(\epsilon\sqrt{-\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)},\end{gathered}\tag{169}$$

and

$$\begin{gathered}q(x,t)=\left\{-\frac{\Delta_2}{b_1-\kappa(\lambda+\nu)}\right.\\ \left.\times\left[1+\coth\left(\epsilon\sqrt{-\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)},\end{gathered}\tag{170}$$

respectively, provided $\Delta_2[b_1-\kappa(\lambda+\nu)]<0,\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

Type 3. Substituting $r=0$, into the algebraic Eqs. (162) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=\frac{6\Delta_2}{b_1-\kappa(\lambda+\nu)},\\ \delta_1=0,\quad \delta_2=\frac{3h\Delta_0}{4m[b_1-\kappa(\lambda+\nu)]},\\ p=\frac{2m\Delta_2}{\Delta_0},\quad s=\frac{3h^2\Delta_0}{32m\Delta_2},\quad h=h,\\ b_2=-\frac{5[b_1-\kappa(\lambda+\nu)]^2}{36\Delta_2},\quad \alpha=\left(1-\frac{1}{2}m\right)\Delta_0,\end{gathered}\tag{171}$$

provided $b-\kappa(\lambda+\nu)\neq 0,\Delta_0\neq 0$ and $\Delta_2\neq 0$.

If we substitute (171) along with (16)–(29) into Eq. (90), then Eq. (158) has the following solutions.

8.1.4. Soliton solutions.

$$q(x,t)=\left\{\frac{6\Delta_2}{b_1-\kappa(\lambda+\nu)}\left[1-\frac{2\operatorname{sech}^2\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}{4-\left[1+\tanh\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{172}$$

$$q(x,t)=\left\{\frac{6\Delta_2}{b_1-\kappa(\lambda+\nu)}\left[1+\frac{2\operatorname{cosech}^2\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}{4-\left[1+\coth\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{173}$$

$$q(x,t)=\left\{\frac{3\Delta_2}{b_1-\kappa(\lambda+\nu)}\left[2-\frac{\operatorname{sech}^2\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}{1+\tanh\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{174}$$

$$q(x,t)=\left\{\frac{3\Delta_2}{b_1-\kappa(\lambda+\nu)}\left[2+\frac{\operatorname{cosech}^2\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}{1+\coth\left(\epsilon\sqrt{\frac{2m\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{175}$$

provided $\Delta_2[b_1-\kappa(\lambda+\nu)]>0,\Delta_0\Delta_2>0$ and $\epsilon=\pm1$.

8.2. Addendum to Kudryashov's Method

According to this method, we balance UU'' with U^4 in Eq. (161), one gets the same relation (146). Now, we will discuss the following cases:

Case 1. If we choose $p = 1$, then $N = 1$. Thus, we deduce that from (31) that Eq. (161) has the same formal solution (113). Substituting (113) along with (32) into Eq. (161), collecting all the coefficients of each power of $[R(\xi)]^{m_1}\,[R'(\xi)]^j$, $(m_1 = 0,1,2,\dots,4,\ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
-2\Delta_0 m\beta_1^2\chi\ln^2 k + 4m^2\beta_1^4 b_2 - \Delta_0\chi\beta_1^2\ln^2 k + \alpha\chi\beta_1^2\ln^2 k = 0,\\
4m^2\beta_1^3[b_1 - \kappa(\lambda+\nu)] + 16m^2\beta_0\beta_1^3 b_2 - 4\Delta_0 m\beta_0\beta_1\chi\ln^2 k = 0,\\
12m^2\beta_0\beta_1^2[b_1 - \kappa(\lambda+\nu)] - \alpha\beta_1^2\ln^2 k + 24m^2\beta_0^2 b_2\beta_1^2 - 4m^2\beta_1^2\Delta_2 + \Delta_0\beta_1^2\ln^2 k = 0,\\
2\Delta_0 m\beta_0\beta_1\ln^2 k + 12m^2\beta_0^2\beta_1[b_1 - \kappa(\lambda+\nu)] + 16m^2\beta_0^3 b_2\beta_1 - 8m^2\beta_0\beta_1\Delta_2 = 0,\\
-4m^2\beta_0^2\Delta_2 + 4m^2\beta_0^3[b_1 - \kappa(\lambda+\nu)] + 4m^2\beta_0^4 b_2 = 0.
\end{gathered}\tag{176}$$

On solving the above algebraic Eqs. (176) by using the Maple, one gets the following results:

$$\begin{gathered}
\beta_0 = -\frac{3\Delta_0\ln^2 k}{2m[b_1 - \kappa(\lambda+\nu)]},\quad \beta_1 = \frac{3\epsilon\Delta_0\sqrt{2\chi}\ln^2 k}{2m[b_1 - \kappa(\lambda+\nu)]},\\
\alpha = (1-2m)\Delta_0,\quad \Delta_2 = -\frac{\Delta_0}{m}\ln^2 k,\quad b_2 = \frac{2m[b_1 - \kappa(\lambda+\nu)]^2}{9\Delta_0\ln^2 k},
\end{gathered}\tag{177}$$

provided $\chi > 0$ and $\epsilon = \pm 1$.

Substituting (177) along with (33) into Eq. (113), one gets the solutions of Eq. (158) in the form:

$$q(x,t) = \left\{-\frac{3\Delta_0\ln^2 k}{m[b_1 - \kappa(\lambda+\nu)]}\left[\frac{1}{2} + \frac{2\epsilon A\sqrt{2\chi}}{4A^2\exp_k[(x-ct)] + \chi\exp_k[-(x-ct)]}\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x+\omega t+\theta_0)},\tag{178}$$

provided $\Delta_0[b_1 - \kappa(\lambda+\nu)] < 0$.

In particular, if we set $\chi = 4A^2$ in (178), then we have the bright soliton solution of Eq. (128) as:

$$q(x,t) = \left\{-\frac{3\Delta_0\ln^2 k}{m[b_1 - \kappa(\lambda+\nu)]}\left[1 + \epsilon\sqrt{2}\operatorname{sech}[(x-ct)\ln k]\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x+\omega t+\theta_0)}.\tag{179}$$

Case 2. If we choose $p = 2$, then $N = 2$. Thus, we deduce that from (31) that Eq. (131) has the same formal solution (119). Substituting (119) along with Eq. (32) into Eq. (131), collecting all the coefficients of each power of $[R(\xi)]^{m_2}$ $[R'(\xi)]^j$, $(m_2 = 0,1,2,\dots,8,\ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
4m^2\beta_2^4 b_2 + 4\alpha\chi\beta_2^2\ln^2 k - 8\Delta_0 m\beta_2^2\chi\ln^2 k - 4\Delta_0\chi\beta_2^2\ln^2 k = 0,\\
-4\Delta_0\chi\beta_1\beta_2\ln^2 k - 14\Delta_0 m\beta_1\beta_2\chi\ln^2 k + 4\alpha\chi\beta_1\beta_2\ln^2 k + 16m^2\beta_1\beta_2^3 b_2 = 0,\\
16m^2\beta_0\beta_2^3 b_2 + \alpha\chi\beta_1^2\ln^2 k + 4m^2\beta_2^3[b_1 - \kappa(\lambda+\nu)] - 4\Delta_0 m\beta_1^2\chi\ln^2 k - 16\Delta_0 m\beta_0\beta_2\chi\ln^2 k + 24m^2\beta_1^2 b_2\beta_2^2\\
-\Delta_0\chi\beta_1^2\ln^2 k = 0,\\
12m^2\beta_1\beta_2^2[b_1 - \kappa(\lambda+\nu)] + 16m^2\beta_1^3 b_2\beta_2 + 48m^2\beta_0\beta_1 b_2\beta_2^2 - 6\Delta_0 m\beta_0\beta_1\chi\ln^2 k = 0,\\
24m^2\beta_0^2 b_2\beta_2^2 + 4m^2\beta_1^4 b_2 + 4\Delta_0\beta_2^2\ln^2 k - 4m^2\beta_2^2\Delta_2 + 12m^2\beta_0\beta_2^2[b_1 - \kappa(\lambda+\nu)]\\
+ 48m^2\beta_0\beta_1^2 b_2\beta_2 - 4\alpha\beta_2^2\ln^2 k + 12m^2\beta_1^2\beta_2[b_1 - \kappa(\lambda+\nu)] = 0,\\
24m^2\beta_0\beta_1[b_1 - \kappa(\lambda+\nu)]\beta_2 - 4\alpha\beta_1\beta_2\ln^2 k + 16m^2\beta_0\beta_1^3 b_2 + 2\Delta_0 m\beta_1\beta_2\ln^2 k - 8m^2\beta_1\beta_2\Delta_2\\
+ 4m^2\beta_1^3[b_1 - \kappa(\lambda+\nu)] + 48m^2\beta_0^2 b_2\beta_1\beta_2 + 4\Delta_0\beta_1\beta_2\ln^2 k = 0,\\
12m^2\beta_0\beta_1^2[b_1 - \kappa(\lambda+\nu)] - 8m^2\beta_0\beta_2\Delta_2 + 8\Delta_0 m\beta_0\beta_2\ln^2 k - 4m^2\beta_1^2\Delta_2 + \Delta_0\beta_1^2\ln^2 k\\
- \alpha\beta_1^2\ln^2 k + 12m^2\beta_0^2\beta_2[b_1 - \kappa(\lambda+\nu)] + 16m^2\beta_0^3 b_2\beta_2 + 24m^2\beta_0^2 b_2\beta_1^2 = 0,\\
2\Delta_0 m\beta_0\beta_1\ln^2 k + 12m^2\beta_0^2\beta_1[b_1 - \kappa(\lambda+\nu)] + 16m^2\beta_0^3 b_2\beta_1 - 8m^2\beta_0\beta_1\Delta_2 = 0,\\
+4m^2\beta_0^4 b_2 + 4m^2\beta_0^3[b_1 - \kappa(\lambda+\nu)] - 4m^2\beta_0^2\Delta_2 = 0.
\end{gathered}\tag{180}$$

On solving the above algebraic Eqs. (180) by using the Maple, one gets the following results:

$$\beta_0 = -\frac{6\Delta_0 \ln^2 k}{m[b_1 - \kappa(\lambda+\nu)]}, \quad \beta_1 = 0, \quad \beta_2 = \frac{6\epsilon\Delta_0\sqrt{2\chi}\ln^2 k}{m[b_1 - \kappa(\lambda+\nu)]}, \quad \alpha = (1-2m)\Delta_0,$$
$$\Delta_2 = -\frac{4\Delta_0}{m}\ln^2 k, \quad b_2 = \frac{m[b_1 - \kappa(\lambda+\nu)]^2}{18\Delta_0 \ln^2 k}, \tag{181}$$

provided $\chi > 0$ and $\epsilon = \pm 1$.

Substituting (181) along with (33) into Eq. (119), one gets the solutions of Eq. (158) in the form:

$$q(x,t) = \left\{-\frac{6\Delta_0 \ln^2 k}{m[b_1 - \kappa(\lambda+\nu)]}\left[1 + \frac{4\epsilon A\sqrt{2\chi}}{4A^2 \exp_k[2(x-ct)] + \chi \exp_k[-2(x-ct)]}\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{182}$$

provided $\Delta_0[b_1 - \kappa(\lambda+\nu)] < 0$.

In particular, if we set $\chi = 4A^2$ in (182), then we have the bright soliton solution of Eq. (158) as:

$$q(x,t) = \left\{-\frac{6\Delta_0 \ln^2 k}{m[b_1 - \kappa(\lambda+\nu)]}\left[1 + \epsilon\sqrt{2}\operatorname{sech}[2(x-ct)\ln k]\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x + \omega t + \theta_0)}. \tag{183}$$

Similarly, we can find many other solutions by choosing other values for p and N.

9. POLYNOMIAL LAW

For the polynomial law nonlinearity, we have

$$F(\phi) = b_1\phi + b_2\phi^2 + b_3\phi^3, \tag{184}$$

where b_1, b_2 and b_3 are arbitrary constants such that $b_3 \neq 0$.

Equation (1) corresponding to polynomial law nonlinearity (184) is given by:

$$iq_t + iaq_{xxx} + \left(b_1|q|^2 + b_2|q|^4 + b_3|q|^6\right)q = \alpha\frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q + i\left[\delta q_x + \lambda\left(|q|^{2m}q\right)_x + \mu\left(|q|^{2m}\right)_x q + \nu|q|^{2m}q_x\right], \tag{185}$$

where Eq. (41) reduces to:

$$\Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2 + b_1\phi^4 - \kappa(\lambda+\nu)\phi^{2m+2} + b_2\phi^6 + b_3\phi^8 = 0. \tag{186}$$

For integrability, one must select $m = 1$. This leads to the modification of Eq. (1) corresponding to polynomial law nonlinearity as:

$$iq_t + iaq_{xxx} + \left(b_1|q|^2 + b_2|q|^4 + b_3|q|^6\right)q = \alpha\frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q + i\left[\delta q_x + \lambda\left(|q|^2 q\right)_x + \mu\left(|q|^2\right)_x q + \nu|q|^2 q_x\right]. \tag{187}$$

Consequently, Eq. (186) becomes:

$$\Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2 + [b_1 - \kappa(\lambda+\nu)]\phi^4 + b_2\phi^6 + b_3\phi^8 = 0. \tag{188}$$

Balancing $\phi\phi''$ and ϕ^8 in Eq. (188), gives the balance number $N = \frac{1}{3}$. Since the balance number is not integer, then we take into consideration the transformation

$$\phi(\xi) = [U(\xi)]^{\frac{1}{3}}, \tag{189}$$

where $U(\xi)$ is a new positive function of ξ. Substituting (189) into (188), we have the new equation

$$3\Delta_0 UU'' - (\alpha + 2\Delta_0)U'^2 - 9\Delta_2 U^2 + 9[b_1 - \kappa(\lambda+\nu)]U^{\frac{8}{3}} + 9b_2 U^{\frac{10}{3}} + 9b_3 U^4 = 0. \tag{190}$$

For integrability, one must select:

$$b_2 = 0, \quad b_1 = \kappa(\lambda+\nu). \tag{191}$$

This leads to the modification of Eq. (1) corresponding to polynomial law nonlinearity as:

$$iq_t + iaq_{xxx} + \left[\kappa(\lambda+\nu)|q|^2 + b_3|q|^6\right]q = \alpha\frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q + i\left[\delta q_x + \lambda\left(|q|^2 q\right)_x + \mu\left(|q|^2\right)_x q + \nu|q|^2 q_x\right]. \tag{192}$$

Consequently, Eq. (190) changes to:

$$3\Delta_0 UU'' - (\alpha + 2\Delta_0)U'^2 - 9\Delta_2 U^2 + 9b_3 U^4 = 0. \tag{193}$$

In the next subsections, we will solve Eq. (193) using the following two methods.

9.1. New Mapping Method

According to the new mapping method, we balance UU'' with U^4 in Eq. (193) yields the balance number $N = 1$. Now, from (6), the solution of Eq. (193) has the same formal solution (90). Substituting (90) along with (7) into Eq. (193), collecting all the coefficients of $F^l(\xi)[F'(\xi)]^j$ $(l = 0,1,\ldots,8,\ j = 0,1)$ and setting them to zero, we have the following algebraic equations:

$$
\begin{gathered}
\frac{16}{3}\Delta_0 s\delta_2^2 + 9b_3\delta_2^4 - \frac{4}{3}\alpha s\delta_2^2 = 0,\\
+\frac{25}{3}\Delta_0 s\delta_1\delta_2 + 36b_3\delta_1\delta_2^3 - \frac{4}{3}\alpha s\delta_1\delta_2 = 0,\\
36b_3\delta_0\delta_2^3 + \frac{7}{3}\Delta_0 s\delta_1^2 + 54b_3\delta_1^2\delta_2^2 + 8\Delta_0\delta_2 s\delta_0\\
-\ 2\alpha h\delta_2^2 + 5\Delta_0 h\delta_2^2 - \frac{1}{3}\alpha s\delta_1^2 = 0,\\
3\Delta_0\delta_1 s\delta_0 + 8\Delta_0 h\delta_1\delta_2 - 2\alpha h\delta_1\delta_2\\
+\ 108b_3\delta_0\delta_1\delta_2^2 + 36b_3\delta_1^3\delta_2 = 0,\\
9\Delta_0\delta_2 h\delta_0 + 9b_3\delta_1^4 - \frac{1}{2}\alpha h\delta_1^2 - 9\Delta_2\delta_2^2\\
+\ 54b_3\delta_0^2\delta_2^2 - 4\alpha p\delta_2^2 + 2\Delta_0 h\delta_1^2\\
+108b_3\delta_0\delta_1^2\delta_2 + 4\Delta_0 p\delta_2^2 = 0,\\
3\Delta_0\delta_1 h\delta_0 + 36b_3\delta_0\delta_1^3 + 7\Delta_0 p\delta_1\delta_2\\
-\ 4\alpha p\delta_1\delta_2 - 18\Delta_2\delta_1\delta_2 + 108b_3\delta_0^2\delta_1\delta_2 = 0,\\
\Delta_0 p\delta_1^2 + 54b_3\delta_0^2\delta_1^2 - 4\alpha r\delta_2^2 + 36b_3\delta_0^3\delta_2\\
-\ \alpha p\delta_1^2 - 18\Delta_2\delta_0\delta_2 - 2\Delta_0 r\delta_2^2\\
-\ 9\Delta_2\delta_1^2 + 12\Delta_0\delta_2 p\delta_0 = 0,\\
-4\alpha r\delta_1\delta_2 + 3\Delta_0\delta_1 p\delta_0 - 2\Delta_0 r\delta_1\delta_2\\
+\ 36b_3\delta_0^3\delta_1 - 18\Delta_2\delta_0\delta_1 = 0,\\
-9\Delta_2\delta_0^2 + 6\Delta_0\delta_2 r\delta_0 - \alpha r\delta_1^2 - 2\Delta_0 r\delta_1^2 + 9b_3\delta_0^4 = 0.
\end{gathered}
\tag{194}
$$

With the aid of the solutions (8)–(29) we have the following types of solutions:

Type 1. Substituting $s = \dfrac{3h^2}{16p}$, $r = \dfrac{16p^2}{27h}$, into the algebraic Eqs. (194) and solve them by Maple, one gets the following results:

$$
\delta_0 = \epsilon\sqrt{-\frac{\Delta_2}{3b_3}},\quad \delta_1 = 0,\quad \delta_2 = \frac{\epsilon h\Delta_0}{\Delta_2}\sqrt{-\frac{\Delta_2}{108b_3}},\quad p = \frac{9\Delta_2}{2\Delta_0},\quad h = h,\quad \alpha = \frac{5}{2}\Delta_0,
\tag{195}
$$

provided $\Delta_2 b_3 < 0, \Delta_0 \neq 0$ and $\epsilon = \pm 1$.

If we substitute (195) along with (8)–(12) into Eq. (90), then Eq. (192) has the following solutions.

9.1.1. Soliton solutions.

$$
q(x,t) = \left\{\sqrt{-\frac{\Delta_2}{3b_3}}\left[1 - \frac{4\tanh^2\left(\epsilon\sqrt{-\frac{3\Delta_2}{\Delta_0}}(x-ct)\right)}{3+\tanh^2\left(\epsilon\sqrt{-\frac{3\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3}} e^{i(-\kappa x+\omega t+\theta_0)},
\tag{196}
$$

and

$$
q(x,t) = \left\{\sqrt{-\frac{\Delta_2}{3b_3}}\left[1 - \frac{4\coth^2\left(\epsilon\sqrt{-\frac{3\Delta_2}{\Delta_0}}(x-ct)\right)}{3+\coth^2\left(\epsilon\sqrt{-\frac{3\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3}} e^{i(-\kappa x+\omega t+\theta_0)},
\tag{197}
$$

provided $\Delta_2 b_3 < 0, \Delta_0\Delta_2 < 0$ and $\epsilon = \pm 1$.

9.1.2. Periodic solutions.

$$
q(x,t) = \left\{\sqrt{-\frac{\Delta_2}{3b_3}}\left[1 + \frac{4\tan^2\left(\epsilon\sqrt{\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)}{3-\tan^2\left(\epsilon\sqrt{\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3}} e^{i(-\kappa x+\omega t+\theta_0)},
\tag{198}
$$

and

$$q(x,t)=\left\{\sqrt{-\frac{\Delta_2}{3b_3}}\left[1+\frac{4\cot^2\left(\epsilon\sqrt{\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)}{3-\cot^2\left(\epsilon\sqrt{\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{199}$$

provided $\Delta_2 b_3<0, \Delta_0\Delta_2>0$ and $\varepsilon=\pm 1$.

Type 2. Substituting $s=\frac{3h^2}{16p}$, $r=0$, into the algebraic Eqs. (194) and solve them by Maple, we get the following results:

$$\delta_0=\epsilon\sqrt{\frac{\Delta_2}{b_3}},\quad \delta_1=0,\quad \delta_2=-\frac{\epsilon h\Delta_0}{6\Delta_2}\sqrt{\frac{\Delta_2}{b_3}},$$
$$p=-\frac{3\Delta_2}{2\Delta_0},\quad h=h,\quad \alpha=\frac{5}{2}\Delta_0, \tag{200}$$

provided $\Delta_2 b_3>0, \Delta_0\neq 0$ and $\varepsilon=\pm 1$.

If we substitute (200) along with (14) and (15) into Eq. (90), then Eq. (192) has the following solutions.

9.1.3. Dark and singular solitons.

$$q(x,t)=\left\{\frac{1}{2}\sqrt{\frac{\Delta_2}{b_3}}\right.$$
$$\left.\times\left[1+\tanh\left(\varepsilon\sqrt{-\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{201}$$

and

$$q(x,t)=\left\{\frac{1}{2}\sqrt{\frac{\Delta_2}{b_3}}\right.$$
$$\left.\times\left[1+\coth\left(\varepsilon\sqrt{-\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{202}$$

respectively, provided $\Delta_2 b_3>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm 1$.

Type 3. Substituting $r=0$, into the algebraic Eqs. (194) and solve them by Maple, we get the following results:

$$\delta_0=\epsilon\sqrt{\frac{\Delta_2}{b_3}},\quad \delta_1=0,\quad \delta_2=-\frac{\epsilon h\Delta_0}{6\Delta_2}\sqrt{\frac{\Delta_2}{b_3}},$$
$$p=-\frac{3\Delta_2}{2\Delta_0},\quad s=-\frac{3h^2\Delta_0}{8\Delta_2},\quad h=h,\quad \alpha=\frac{5}{2}\Delta_0, \tag{203}$$

provided $\Delta_2 b_3>0, \Delta_0\neq 0$ and $\epsilon=\pm 1$.

If we substitute (203) along with (16)–(29) into Eq. (90), then Eq. (192) has the following solutions.

9.1.4. Soliton solutions.

$$q(x,t)=\left\{\sqrt{\frac{\Delta_2}{b_3}}\left[1-\frac{2\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)}{4-\left[1+\tanh\left(\epsilon\sqrt{-\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{204}$$

$$q(x,t)=\left\{\sqrt{\frac{\Delta_2}{b_3}}\left[1+\frac{2\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{3\Delta_2}{\Delta_0}}(x-ct)\right)}{4-\left[1+\coth\left(\epsilon\sqrt{-\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{205}$$

$$q(x,t)=\left\{\frac{1}{2}\sqrt{\frac{\Delta_2}{b_3}}\left[2-\frac{\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)}{1+\tanh\left(\epsilon\sqrt{-\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{206}$$

$$q(x,t)=\left\{\frac{1}{2}\sqrt{\frac{\Delta_2}{b_3}}\left[2+\frac{\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)}{1+\coth\left(\epsilon\sqrt{-\frac{3\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{207}$$

provided $\Delta_2 b_3>0, \Delta_0\Delta_2<0$ and $\epsilon=\pm 1$.

9.2. Addendum to Kudryashov's Method

According to this method, we balance UU'' with U^4 in Eq. (131), the same relation (146). Now, we will discuss the following cases:

Case 1. If we choose $p=1$, then $N=1$. Thus, we deduce that from (31) that Eq. (193) has the same formal solution (113). Substituting (113) along with (32) into Eq. (193), collecting all the coefficients of each power of $[R(\xi)]^{m_1}\,[R'(\xi)]^j,\ (m_1=0,1,2,\ldots,4,\ j=0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}-4\Delta_0\ln^2 k\chi\beta_1^2+\alpha\ln^2 k\chi\beta_1^2+9b_3\beta_1^4=0,\quad -6\Delta_0\ln^2 k\beta_1\chi\beta_0+36b_3\beta_0\beta_1^3=0,\\ \Delta_0\ln^2 k\beta_1^2-\alpha\ln^2 k\beta_1^2+54b_3\beta_0^2\beta_1^2-9\Delta_2\beta_1^2=0,\\ -18\Delta_2\beta_0\beta_1+36b_3\beta_0^3\beta_1+3\Delta_0\ln^2 k\beta_1\beta_0=0,\quad 9b_3\beta_0^4-9\Delta_2\beta_0^2=0.\end{gathered}\tag{208}$$

On solving the above algebraic Eqs. (208) by using the Maple, one gets the following results:

$$\beta_0=0,\quad \beta_1=\epsilon\sqrt{\frac{\left(3\Delta_2+\Delta_0\ln^2 k\right)\chi}{3b_3}},\quad \alpha=\frac{-9\Delta_2+\Delta_0\ln^2 k}{\ln^2 k},\tag{209}$$

provided $\left(3\Delta_2+\Delta_0\ln^2 k\right)\chi b_3>0$ and $\epsilon=\pm1$.

Substituting (209) along with (33) into Eq. (193), one gets the solutions of Eq. (192) in the form:

$$q(x,t)=\left\{\sqrt{\frac{\left(3\Delta_2+\Delta_0\ln^2 k\right)\chi}{3b_3}}\left[\frac{4A}{4A^2\exp_k[(x-ct)]+\chi\exp_k[-(x-ct)]}\right]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}.\tag{210}$$

In particular, if we set $\chi=4A^2$ in (210), then we have the bright soliton solution of Eq. (192) as:

$$q(x,t)=\left\{\sqrt{\frac{3\Delta_2+\Delta_0\ln^2 k}{3b_3}}\operatorname{sech}[(x-ct)\ln k]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{211}$$

while, if $\chi=-4A^2$, one gets the singular soliton solution of Eq. (192) as:

$$q(x,t)=\left\{\sqrt{-\frac{3\Delta_2+\Delta_0\ln^2 k}{3b_3}}\ \operatorname{cosech}[(x-ct)\ln k]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}.\tag{212}$$

Case 2. If we choose $p=2$, then $N=2$. Thus, we deduce that from (31) that Eq. (193) has the same formal solution (119). Substituting (119) along with Eq. (32) into Eq. (193), collecting all the coefficients of each power of $[R(\xi)]^{m_2}\,[R'(\xi)]^j,\ (m_2=0,1,2,\ldots,8,\ j=0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}9b_3\beta_2^4-16\Delta_0\ln^2 k\chi\beta_2^2+4\alpha\ln^2 k\chi\beta_2^2=0,\\ 4\alpha\ln^2 k\chi\beta_1\beta_2-25\Delta_0\ln^2 k\chi\beta_1\beta_2+36b_3\beta_1\beta_2^3=0,\\ 36b_3\beta_0\beta_2^3-7\Delta_0\ln^2 k\chi\beta_1^2+\alpha\ln^2 k\chi\beta_1^2\\ -24\Delta_0\ln^2 k\beta_2\chi\beta_0+54b_3\beta_1^2\beta_2^2=0,\\ 108b_3\beta_0\beta_1\beta_2^2+36b_3\beta_1^3\beta_2-9\Delta_0\ln^2 k\beta_1\chi\beta_0=0,\\ 4\Delta_0\ln^2 k\beta_2^2+54b_3\beta_0^2\beta_2^2-4\alpha\ln^2 k\beta_2^2\\ +9b_3\beta_1^4+108b_3\beta_0\beta_1^2\beta_2-9\Delta_2\beta_2^2=0,\\ 108b_3\beta_0^2\beta_1\beta_2-4\alpha\ln^2 k\beta_1\beta_2+36b_3\beta_0\beta_1^3\\ +7\Delta_0\ln^2 k\beta_1\beta_2-18\Delta_2\beta_1\beta_2=0,\\ 12\Delta_0\ln^2 k\beta_2\beta_0-\alpha\ln^2 k\beta_1^2\\ -9\Delta_2\beta_1^2+54b_3\beta_0^2\beta_1^2+36b_3\beta_0^3\beta_2\\ -18\Delta_2\beta_0\beta_2+\Delta_0\ln^2 k\beta_1^2=0,\\ -18\Delta_2\beta_0\beta_1+36b_3\beta_0^3\beta_1+3\Delta_0\ln^2 k\beta_1\beta_0=0,\\ -9\Delta_2\beta_0^2+9b_3\beta_0^4=0.\end{gathered}\tag{213}$$

On solving the above algebraic Eqs. (213) by using the Maple, one gets the following results:

$$\beta_0=0,\quad \beta_1=0,\quad \beta_2=\epsilon\sqrt{\frac{\left(3\Delta_2+4\Delta_0\ln^2 k\right)\chi}{3b_3}},\quad \alpha=\frac{-9\Delta_2+4\Delta_0\ln^2 k}{4\ln^2 k},\tag{214}$$

provided

$$\left(3\Delta_2+4\Delta_0\ln^2 k\right)\chi b_3>0\ \text{and}\ \epsilon=\pm1.\tag{215}$$

Substituting (214) along with (33) into Eq. (193), one gets the solutions of Eq. (192) in the form:

$$q(x,t)=\left\{\sqrt{\frac{\left(3\Delta_2+4\Delta_0\ln^2 k\right)\chi}{3b_3}}\left[\frac{4A}{4A^2\exp_k\left[2(x-ct)\right]+\chi\exp_k\left[-2(x-ct)\right]}\right]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{216}$$

In particular, if we set $\chi=4A^2$ in (216), then we have the bright soliton solution of Eq. (192) as:

$$q(x,t)=\left\{\sqrt{\frac{3\Delta_2+4\Delta_0\ln^2 k}{3b_3}}\operatorname{sech}\left[2(x-ct)\ln k\right]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{217}$$

while, if $\chi=-4A^2$, one gets the singular soliton solution of Eq. (192) as:

$$q(x,t)=\left\{\sqrt{-\frac{3\Delta_2+4\Delta_0\ln^2 k}{3b_3}}\operatorname{cosech}\left[2(x-ct)\ln k\right]\right\}^{\frac{1}{3}}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{218}$$

Similarly, we can find many other solutions by choosing other values for p and N.

10. TRIPLE-POWER LAW

For the triple-power law nonlinearity, we have

$$F(\phi)=b_1\phi^n+b_2\phi^{2n}+b_3\phi^{3n}, \tag{219}$$

where b_1,b_2 and b_3 are arbitrary constants such that $b_3\neq 0$. Equation (1) corresponding to triple-power law nonlinearity (219) is given by:

$$\begin{aligned}&iq_t+iaq_{xxx}+\left(b_1|q|^{2n}+b_2|q|^{4n}+b_3|q|^{6n}\right)q=\alpha\frac{|q_x|^2}{q^*}\\&+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q\\&+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_xq+\nu|q|^{2m}q_x\right],\end{aligned} \tag{220}$$

where Eq. (41) reduces to:

$$\begin{aligned}&\Delta_0\phi\phi''-\alpha\phi'^2-\Delta_2\phi^2+b_1\phi^{2n+2}\\&-\kappa(\lambda+\nu)\phi^{2m+2}+b_2\phi^{4n+2}+b_3\phi^{6n+2}=0.\end{aligned} \tag{221}$$

For integrability, one must select $n=m$. This leads to the modification of Eq. (1) corresponding to triple-power law nonlinearity as:

$$\begin{aligned}&iq_t+iaq_{xxx}+\left(b_1|q|^{2m}+b_2|q|^{4m}+b_3|q|^{6m}\right)q=\alpha\frac{|q_x|^2}{q^*}\\&+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q\\&+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_xq+\nu|q|^{2m}q_x\right].\end{aligned} \tag{222}$$

Consequently, Eq. (221) becomes:

$$\begin{aligned}&\Delta_0\phi\phi''-\alpha\phi'^2-\Delta_2\phi^2+\left[b_1-\kappa(\lambda+\nu)\right]\phi^{2m+2}\\&+b_2\phi^{4m+2}+b_3\phi^{6m+2}=0.\end{aligned} \tag{223}$$

Balancing $\phi\phi''$ and ϕ^{6m+2} in Eq. (223), gives the balance number $N=\frac{1}{3m}$. Since the balance number is not integer, then we take into consideration the transformation

$$\phi(\xi)=[U(\xi)]^{\frac{1}{3m}} \tag{224}$$

where $U(\xi)$ is a new positive function of ξ. Substituting (224) into (223), we have the new equation

$$\begin{aligned}&3m\Delta_0UU''-\left[\alpha+(3m-1)\Delta_0\right]U'^2\\&-9m^2\Delta_2U^2+9m^2\left[b_1-\kappa(\lambda+\nu)\right]U^{\frac{8}{3}}\\&+9m^2b_2U^{\frac{10}{3}}+9m^2b_3U^4=0.\end{aligned} \tag{225}$$

For integrability, one must select:

$$b_2=0,\quad b_1=\kappa(\lambda+\nu). \tag{226}$$

This leads to the modification of Eq. (1) corresponding to triple-power law nonlinearity as:

$$\begin{aligned}&iq_t+iaq_{xxx}+\left[\kappa(\lambda+\nu)|q|^{2m}+b_3|q|^{6m}\right]q=\alpha\frac{|q_x|^2}{q^*}\\&+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q\\&+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_xq+\nu|q|^{2m}q_x\right].\end{aligned} \tag{227}$$

Consequently, Eq. (225) changes to:

$$\begin{aligned}&3m\Delta_0UU''-\left[\alpha+(3m-1)\Delta_0\right]U'^2\\&-9m^2\Delta_2U^2+9m^2b_3U^4=0.\end{aligned} \tag{228}$$

In the next subsections, we will solve Eq. (228) using the following two methods.

10.1. New Mapping Method

According to the new mapping method, we balance UU'' with U^4 in Eq. (228) yields the balance number $N = 1$. Now, from (6), the solution of Eq. (228) has the same formal solution (90). Substituting (90) along with (7) into Eq. (228), collecting all the coefficients of $F^l(\xi)[F'(\xi)]^j$ $(l = 0,1,\ldots,8,\ j = 0,1)$ and setting them to zero, we have the following algebraic equations:

$$\begin{gathered}
-4\Delta_0 m\delta_2^2 s - 9m^2\delta_2^4 b_3 + \frac{4}{3}\alpha s\delta_2^2 - \frac{4}{3}\Delta_0 s\delta_2^2 = 0,\\
-7\Delta_0 m\delta_1\delta_2 s - \frac{4}{3}\Delta_0 s\delta_1\delta_2 + \frac{4}{3}\alpha s\delta_1\delta_2 - 36m^2\delta_1\delta_2^3 b_3 = 0,\\
-36m^2\delta_0\delta_2^3 b_3 - 3\Delta_0 m\delta_2^2 h + 2\alpha h\delta_2^2 - 54m^2\delta_1^2 b_3\delta_2^2 + \frac{1}{3}\alpha s\delta_1^2 - \frac{1}{3}\Delta_0 s\delta_1^2\\
-8\Delta_0 m\delta_0\delta_2 s - 2\Delta_0 h\delta_2^2 - 2\Delta_0 m\delta_1^2 s = 0, \quad -6\Delta_0 m\delta_1\delta_2 h - 36m^2\delta_1^3 b_3\delta_2 - 3\Delta_0 m\delta_0\delta_1 s\\
-2\Delta_0 h\delta_1\delta_2 + 2\alpha h\delta_1\delta_2 - 108m^2\delta_0\delta_1 b_3\delta_2^2 = 0,\\
\frac{1}{2}\alpha h\delta_1^2 + 9m^2\delta_2^2\Delta_2 - \frac{1}{2}\Delta_0 h\delta_1^2 - 9\Delta_0 m\delta_0\delta_2 h - 4\Delta_0 p\delta_2^2 - 9m^2\delta_1^4 b_3 - \frac{3}{2}\Delta_0 m\delta_1^2 h + 4\alpha p\delta_2^2\\
-108m^2\delta_0\delta_1^2 b_3\delta_2 - 54m^2\delta_0^2 b_3\delta_2^2 = 0,\\
-108m^2\delta_0^2 b_3\delta_1\delta_2 + 4\alpha p\delta_1\delta_2 - 36m^2\delta_0\delta_1^3 b_3 - 3\Delta_0 m\delta_0\delta_1 h - 3\Delta_0 m\delta_1\delta_2 p + 18m^2\delta_1\delta_2\Delta_2 - 4\Delta_0 p\delta_1\delta_2 = 0,\\
18m^2\delta_0\delta_2\Delta_2 + \alpha p\delta_1^2 - 4\Delta_0 r\delta_2^2 + 4\alpha r\delta_2^2 + 9m^2\delta_1^2\Delta_2 - 12\Delta_0 m\delta_0\delta_2 p - 54m^2\delta_0^2 b_3\delta_1^2\\
-\Delta_0 p\delta_1^2 - 36m^2\delta_0^3 b_3\delta_2 + 6\Delta_0 m\delta_2^2 r = 0,\\
4\alpha r\delta_1\delta_2 - 4\Delta_0 r\delta_1\delta_2 - 36m^2\delta_0^3 b_3\delta_1 + 18m^2\delta_0\delta_1\Delta_2 + 6\Delta_0 m\delta_1\delta_2 r - 3\Delta_0 m\delta_0\delta_1 p = 0,\\
+9m^2\delta_0^2\Delta_2 - \Delta_0 r\delta_1^2 + \alpha r\delta_1^2 - 9m^2\delta_0^4 b_3 + 3\Delta_0 mr\delta_1^2 - 6\Delta_0 m\delta_0\delta_2 r = 0.
\end{gathered} \tag{229}$$

With the aid of the solutions (8)–(29), we have the following types of solutions:

Type 1. Substituting $s = \dfrac{3h^2}{16p}$, $r = \dfrac{16p^2}{27h}$, into the algebraic Eqs. (229) and solve them by Maple, one gets the following results:

$$\begin{gathered}
\delta_0 = \epsilon\sqrt{-\frac{\Delta_2}{3b_3}}, \quad \delta_1 = 0, \quad \delta_2 = \frac{\epsilon h\Delta_0}{m\Delta_2}\sqrt{-\frac{\Delta_2}{108b_3}},\\
p = \frac{9m\Delta_2}{2\Delta_0}, \quad h = h, \quad \alpha = \frac{1}{2}(3m+2)\Delta_0,
\end{gathered} \tag{230}$$

provided $\Delta_2 b_3 < 0$, $\Delta_0 \neq 0$ and $\epsilon = \pm 1$.

If we substitute (230) along with (8)–(12) into Eq. (90), then Eq. (227) has the following solutions.

10.1.1. Soliton solutions.

$$q(x,t) = \left\{\sqrt{-\frac{\Delta_2}{3b_3}}\left[1 - \frac{4\tanh^2\left(\epsilon\sqrt{-\frac{3m\Delta_2}{\Delta_0}}(x-ct)\right)}{3+\tanh^2\left(\epsilon\sqrt{-\frac{3m\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{231}$$

and

$$q(x,t) = \left\{\sqrt{-\frac{\Delta_2}{3b_3}}\left[1 - \frac{4\coth^2\left(\epsilon\sqrt{-\frac{3m\Delta_2}{\Delta_0}}(x-ct)\right)}{3+\coth^2\left(\epsilon\sqrt{-\frac{3m\Delta_2}{\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{232}$$

provided $\Delta_2 b_3 < 0$, $\Delta_0\Delta_2 < 0$ and $\epsilon = \pm 1$.

10.1.2. Periodic solutions.

$$q(x,t)=\left\{\sqrt{-\frac{\Delta_2}{3b_3}}\left[1+\frac{4\tan^2\left(\epsilon\sqrt{\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)}{3-\tan^2\left(\epsilon\sqrt{\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{233}$$

or

$$q(x,t)=\left\{\sqrt{-\frac{\Delta_2}{3b_3}}\left[1+\frac{4\cot^2\left(\epsilon\sqrt{\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)}{3-\cot^2\left(\epsilon\sqrt{\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{234}$$

provided $\Delta_2 b_3 < 0$, $\Delta_0\Delta_2 > 0$ and $\epsilon = \pm 1$.

Type 2. Substituting $s = \frac{3h^2}{16p}$, $r = 0$, into the algebraic Eqs. (229) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=\epsilon\sqrt{\frac{\Delta_2}{b_3}},\quad \delta_1=0,\quad \delta_2=-\frac{\epsilon h\Delta_0}{6m\Delta_2}\sqrt{\frac{\Delta_2}{b_3}},\\ p=-\frac{3m\Delta_2}{2\Delta_0},\quad h=h,\quad \alpha=\frac{1}{2}(3m+2)\Delta_0,\end{gathered} \tag{235}$$

provided $\Delta_2 b_3 > 0$, $\Delta_0 \neq 0$ and $\epsilon = \pm 1$.

If we substitute (235) along with (14) and (15) into Eq. (90), then Eq. (227) has the following solutions:

10.1.3. Dark and singular solitons.

$$\begin{aligned}q(x,t)=&\left\{\frac{1}{2}\sqrt{\frac{\Delta_2}{b_3}}\right.\\ &\times\left.\left[1+\tanh\left(\epsilon\sqrt{-\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{3m}} e^{i(-\kappa x+\omega t+\theta_0)},\end{aligned} \tag{236}$$

and

$$\begin{aligned}q(x,t)=&\left\{\frac{1}{2}\sqrt{\frac{\Delta_2}{b_3}}\right.\\ &\times\left.\left[1+\coth\left(\epsilon\sqrt{-\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{3m}} e^{i(-\kappa x+\omega t+\theta_0)},\end{aligned} \tag{237}$$

respectively, provided $\Delta_2 b_3 > 0$, $\Delta_0\Delta_2 < 0$ and $\epsilon = \pm 1$.

Type 3. Substituting $r = 0$, into the algebraic equations (229) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=\epsilon\sqrt{\frac{\Delta_2}{b_3}},\quad \delta_1=0,\quad \delta_2=-\frac{\epsilon h\Delta_0}{6m\Delta_2}\sqrt{\frac{\Delta_2}{b_3}},\\ p=-\frac{3m\Delta_2}{2\Delta_0},\quad s=-\frac{3h^2\Delta_0}{8m\Delta_2},\quad h=h,\\ \frac{1}{2}(3m+2)\Delta_0,\end{gathered} \tag{238}$$

provided $\Delta_2 b_3 > 0$, $\Delta_0 \neq 0$ and $\epsilon = \pm 1$.

If we substitute (238) along with (16)–(29) into Eq. (90), then Eq. (227) has the following solutions.

10.1.4. Soliton solutions.

$$q(x,t)=\left\{\sqrt{\frac{\Delta_2}{b_3}}\left[1-\frac{2\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)}{4-\left[1+\tanh\left(\epsilon\sqrt{-\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{3m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{239}$$

$$q(x,t)=\left\{\sqrt{\frac{\Delta_2}{b_3}}\left[1+\frac{2\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)}{4-\left[1+\coth\left(\epsilon\sqrt{-\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{3m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{240}$$

$$q(x,t)=\left\{\frac{1}{2}\sqrt{\frac{\Delta_2}{b_3}}\left[2-\frac{\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)}{1+\tanh\left(\epsilon\sqrt{-\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{241}$$

$$q(x,t)=\left\{\frac{1}{2}\sqrt{\frac{\Delta_2}{b_3}}\left[2+\frac{\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)}{1+\coth\left(\epsilon\sqrt{-\frac{3m\Delta_2}{2\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{3m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{242}$$

provided $\Delta_2 b_3>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm 1$.

10.2. Addendim to Kudryashov's Method

According to this method, we balance UU'' with U^4 in Eq. (228), the same relation (146). Now, we will discuss the following cases:

Case 1. If we choose $p=1$, then $N=1$. Thus, we deduce that from (31) that Eq. (228) has the same formal solution (113). Substituting (113) along with (32) into Eq. (228), collecting all the coefficients of each power of $[R(\xi)]^{m_1}\ [R'(\xi)]^j$, $(m_1=0,1,2,\ldots,4,\ j=0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}3\Delta_0\beta_1^2m\chi\ln^2k+\beta_1^2\Delta_0\chi\ln^2k-9m^2\beta_1^4b_3-\beta_1^2\alpha\chi\ln^2k=0,\\ 6\Delta_0\beta_1m\beta_0\chi\ln^2k-36m^2\beta_0\beta_1^3b_3=0,\\ 9m^2\beta_1^2\Delta_2+\beta_1^2\alpha\ln^2k-\beta_1^2\Delta_0\ln^2k-54m^2\beta_0^2b_3\beta_1^2=0,\\ -3\Delta_0\beta_1m\beta_0\ln^2k-36m^2\beta_0^3b_3\beta_1+18m^2\beta_0\beta_1\Delta_2=0,\\ 9m^2\beta_0^2\Delta_2-9m^2\beta_0^4b_3=0.\end{gathered} \tag{243}$$

On solving the above algebraic Eqs. (243) by using the Maple, one gets the following results:

$$\beta_0=0,\quad \beta_1=\epsilon\sqrt{\frac{\left(3m\Delta_2+\Delta_0\ln^2k\right)\chi}{3mb_3}},\quad \alpha=\frac{-9m^2\Delta_2+\Delta_0\ln^2k}{\ln^2k}, \tag{244}$$

provided $\left(3m\Delta_2+\Delta_0\ln^2k\right)\chi b_3>0$ and $\epsilon=\pm 1$.

Substituting (244) along with (33) into Eq. (228), one gets the solutions of Eq. (227) in the form:

$$q(x,t)=\left\{\sqrt{\frac{\left(3m\Delta_2+\Delta_0\ln^2k\right)\chi}{3mb_3}}\left[\frac{4A}{4A^2\exp_k[(x-ct)]+\chi\exp_k[-(x-ct)]}\right]\right\}^{\frac{1}{3m}}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{245}$$

In particular, if we set $\chi=4A^2$ in (245), then we have the bright soliton solution of Eq. (227) as:

$$q(x,t)=\left\{\sqrt{\frac{3m\Delta_2+\Delta_0\ln^2k}{3mb_3}}\operatorname{sech}[(x-ct)\ln k]\right\}^{\frac{1}{3m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{246}$$

while, if $\chi=-4A^2$, one gets the singular soliton solution of Eq. (227) as:

$$q(x,t)=\left\{\sqrt{-\frac{3m\Delta_2+\Delta_0\ln^2k}{3mb_3}}\operatorname{cosech}[(x-ct)\ln k]\right\}^{\frac{1}{3m}}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{247}$$

Case 2. If we choose $p = 2$, then $N = 2$. Thus, we deduce that from (31) that Eq. (228) has the same formal solution (119). Substituting (119) along with Eq. (32) into Eq. (228), collecting all the coefficients of each power of $[R(\xi)]^{m_2}\,[R'(\xi)]^{j}$, $(m_2 = 0,1,2,\dots,8,\ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
4\Delta_0\chi\beta_2^2\ln^2 k - 9m^2\beta_2^4 b_3 - 4\alpha\chi\beta_2^2\ln^2 k + 12\Delta_0 m\beta_2^2\chi\ln^2 k = 0,\\
21\Delta_0 m\beta_1\beta_2\chi\ln^2 k - 4\alpha\chi\beta_1\beta_2\ln^2 k - 36m^2\beta_1\beta_2^3 b_3 + 4\Delta_0\chi\beta_1\beta_2\ln^2 k = 0,\\
24\Delta_0 m\beta_0\beta_2\chi\ln^2 k - \beta_1^2\alpha\chi\ln^2 k + 6\Delta_0\beta_1^2 m\chi\ln^2 k - 54m^2\beta_1^2 b_3\beta_2^2\\
-\,36m^2\beta_0\beta_2^3 b_3 + \beta_1^2\Delta_0\chi\ln^2 k = 0,\\
-36m^2\beta_1^3 b_3\beta_2 - 108m^2\beta_0\beta_1 b_3\beta_2^2 + 9\Delta_0\beta_1 m\beta_0\chi\ln^2 k = 0,\\
-54m^2\beta_0^2 b_3\beta_2^2 - 9m^2\beta_1^4 b_3 - 4\Delta_0\beta_2^2\ln^2 k - 108m^2\beta_0\beta_1^2 b_3\beta_2 + 9m^2\beta_2^2\Delta_2 + 4\alpha\beta_2^2\ln^2 k = 0,\\
-4\Delta_0\beta_1\beta_2\ln^2 k - 36m^2\beta_0\beta_1^3 b_3 - 108m^2\beta_0^2 b_3\beta_1\beta_2 - 3\Delta_0 m\beta_1\beta_2\ln^2 k\\
+18m^2\beta_1\beta_2\Delta_2 + 4\alpha\beta_1\beta_2\ln^2 k = 0,\\
-54m^2\beta_0^2 b_3\beta_1^2 - 36m^2\beta_0^3 b_3\beta_2 + 18m^2\beta_0\beta_2\Delta_2 - \beta_1^2\Delta_0\ln^2 k - 12\Delta_0 m\beta_0\beta_2\ln^2 k + 9m^2\beta_1^2\Delta_2 + \beta_1^2\alpha\ln^2 k = 0,\\
-3\Delta_0\beta_1 m\beta_0\ln^2 k - 36m^2\beta_0^3 b_3\beta_1 + 18m^2\beta_0\beta_1\Delta_2 = 0,\\
9m^2\beta_0^2\Delta_2 - 9m^2\beta_0^4 b_3 = 0.
\end{gathered}\tag{248}$$

On solving the above algebraic Eqs. (248) by using the Maple, one gets the following results:

$$\beta_0 = 0,\quad \beta_1 = 0,\quad \beta_2 = \epsilon\sqrt{\frac{\left(3m\Delta_2 + 4\Delta_0\ln^2 k\right)\chi}{3mb_3}},\qquad \alpha = \frac{-9m^2\Delta_2 + 4\Delta_0\ln^2 k}{4\ln^2 k},\tag{249}$$

provided $\left(3m\Delta_2 + 4\Delta_0\ln^2 k\right)\chi b_3 > 0$ and $\epsilon = \pm 1$.

Substituting (249) along with (33) into Eq. (228), one gets the solutions of Eq. (227) in the form:

$$q(x,t) = \left\{\sqrt{\frac{\left(3m\Delta_2 + 4\Delta_0\ln^2 k\right)\chi}{3mb_3}\left[\frac{4A}{4A^2\exp_k\left[2(x-ct)\right] + \chi\exp_k\left[-2(x-ct)\right]}\right]}\right\}^{\frac{1}{3m}} e^{i(-\kappa x+\omega t+\theta_0)}.\tag{250}$$

In particular, if we set $\chi = 4A^2$ in (250), then we have the bright soliton solution of Eq. (227) as:

$$q(x,t) = \left\{\sqrt{\frac{3m\Delta_2 + 4\Delta_0\ln^2 k}{3mb_3}}\operatorname{sech}\left[2(x-ct)\ln k\right]\right\}^{\frac{1}{3m}} e^{i(-\kappa x+\omega t+\theta_0)},\tag{251}$$

while, if $\chi = -4A^2$, one gets the singular soliton solution of Eq. (227) as:

$$q(x,t) = \left\{\sqrt{-\frac{3m\Delta_2 + 4\Delta_0\ln^2 k}{3mb_3}}\operatorname{cosech}\left[2(x-ct)\ln k\right]\right\}^{\frac{1}{3m}} e^{i(-\kappa x+\omega t+\theta_0)}.\tag{252}$$

Similarly, we can find many other solutions by choosing other values for p and N.

11. ANTI-CUBIC LAW

For the anti-cubic law nonlinearity, we have

$$F(\phi) = b_1\phi + b_2\phi^2 + \frac{b_3}{\phi^2},\tag{253}$$

where b_3 is the coefficient of anti-cubic nonlinearity, b_1 is the Kerr law nonlinearity and b_2 is the pseudo-quintic nonlinear coefficient.

Equation (1) corresponding to anti-cubic law nonlinearity (253) is given by:

$$iq_t + iaq_{xxx} + \left(b_1|q|^2 + b_2|q|^4 + \frac{b_3}{|q|^4}\right)q = \alpha\frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q + i\left[\delta q_x + \lambda\left(|q|^{2m}q\right)_x + \mu\left(|q|^{2m}\right)_x q + \nu|q|^{2m}q_x\right], \quad (254)$$

where Eq. (41) reduces to:

$$\Delta_0\phi^3\phi'' - \alpha\phi^2\phi'^2 + b_3 - \Delta_2\phi^4 + b_1\phi^6 - \kappa(\lambda+\nu)\phi^{2m+4} + b_2\phi^8 = 0. \quad (255)$$

For integrability, one must select $m = 1$. This leads to the modification of Eq. (1) corresponding to anti-cubic law nonlinearity as:

$$iq_t + iaq_{xxx} + \left(b_1|q|^2 + b_2|q|^4 + \frac{b_3}{|q|^4}\right)q = \alpha\frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q + i\left[\delta q_x + \lambda\left(|q|^2 q\right)_x + \mu\left(|q|^2\right)_x q + \nu|q|^2 q_x\right]. \quad (256)$$

Consequently, Eq. (255) changes to:

$$\Delta_0\phi^3\phi'' - \alpha\phi^2\phi'^2 + b_3 - \Delta_2\phi^4 + [b_1 - \kappa(\lambda+\nu)]\phi^6 + b_2\phi^8 = 0. \quad (257)$$

Balancing $\phi^3\phi''$ and φ^8 in Eq. (257), gives the balance number $N = \frac{1}{2}$. Since the balance number N is not integer, then we take into consideration the transformation

$$\phi(\xi) = [U(\xi)]^{\frac{1}{2}}, \quad (258)$$

where $U(\xi)$ is a new positive function of ξ. Substituting (258) into (257), we have the new equation

$$2\Delta_0 UU'' - (\alpha + \Delta_0)U'^2 + 4b_3 - 4\Delta_2 U^2 + 4[b_1 - \kappa(\lambda+\nu)]U^3 + 4b_2 U^4 = 0. \quad (259)$$

In the next two subsections, we will solve Eq. (259) using the following two methods.

11.1. New Mapping Method

According to the new mapping method, we balance UU'' with U^4 in Eq. (259) yields the balance number $N = 1$. Now, from (6), the solution of Eq. (259) has the same formal solution (90). Substituting (90) along with (7) into Eq. (259), collecting all the coefficients of $F^l(\xi)[F'(\xi)]^j$ $(l = 0,1,\ldots,8,\ j = 0,1)$ and setting them to zero, we have the following algebraic equations:

$$\begin{aligned}
&\frac{4}{3}\alpha s\delta_2^2 - 4b_2\delta_2^4 - 4\Delta_0\delta_2^2 s = 0,\\
&-16b_2\delta_1\delta_2^3 + \frac{4}{3}\alpha s\delta_1\delta_2 - 6\Delta_0\delta_1\delta_2 s = 0,\\
&-4[b_1 - \kappa(\lambda+\nu)]\delta_2^3 + \frac{1}{3}\alpha s\delta_1^2 - 24b_2\delta_1^2\delta_2^2 - \frac{16}{3}\Delta_0\delta_0\delta_2 s - 4\Delta_0\delta_2^2 h - \frac{5}{3}\Delta_0\delta_1^2 s + 2\alpha h\delta_2^2 - 16b_2\delta_0\delta_2^3 = 0,\\
&-16b_2\delta_1^3\delta_2 + 2\alpha h\delta_1\delta_2 - 48b_2\delta_0\delta_1\delta_2^2 - 6\Delta_0\delta_1\delta_2 h - 2\Delta_0\delta_0\delta_1 s - 12[b_1 - \kappa(\lambda+\nu)]\delta_1\delta_2^2 = 0,\\
&\frac{1}{2}\alpha h\delta_1^2 - 4b_2\delta_1^4 - 24b_2\delta_0^2\delta_2^2 + 4\alpha p\delta_2^2 - 12[b_1 - \kappa(\lambda+\nu)]\delta_0\delta_2^2 - 12[b_1 - \kappa(\lambda+\nu)]\delta_1^2\delta_2\\
&\quad - 6\Delta_0\delta_0\delta_2 h - 4\Delta_0\delta_2^2 p - 48b_2\delta_0\delta_1^2\delta_2 - \frac{3}{2}\Delta_0\delta_1^2 h + 4\Delta_2\delta_2^2 = 0,\\
&-2\Delta_0\delta_0\delta_1 h - 24[b_1 - \kappa(\lambda+\nu)]\delta_0\delta_1\delta_2 + 4\alpha p\delta_1\delta_2 - 48b_2\delta_0^2\delta_1\delta_2 - 6\Delta_0\delta_1\delta_2 p\\
&\quad + 8\Delta_2\delta_1\delta_2 - 16b_2\delta_0\delta_1^3 - 4[b_1 - \kappa(\lambda+\nu)]\delta_1^3 = 0,\\
&8\Delta_2\delta_0\delta_2 - \Delta_0\delta_1^2 p - 16b_2\delta_0^3\delta_2 + \alpha p\delta_1^2 - 8\Delta_0\delta_0\delta_2 p - 12[b_1 - \kappa(\lambda+\nu)]\delta_0\delta_1^2 - 24b_2\delta_0^2\delta_1^2\\
&\quad - 12[b_1 - \kappa(\lambda+\nu)]\delta_0^2\delta_2 + 4\alpha r\delta_2^2 + 4\Delta_2\delta_1^2 = 0,\\
&-12[b_1 - \kappa(\lambda+\nu)]\delta_0^2\delta_1 + 4\alpha r\delta_1\delta_2 - 2\Delta_0\delta_0\delta_1 p + 8\Delta_2\delta_0\delta_1 - 16b_2\delta_0^3\delta_1 = 0,\\
&-4b_3 + 4\Delta_2\delta_0^2 - 4\Delta_0\delta_0\delta_2 r - 4[b_1 - \kappa(\lambda+\nu)]\delta_0^3 - 4b_2\delta_0^4 + \Delta_0 r\delta_1^2 + \alpha r\delta_1^2 = 0.
\end{aligned} \quad (260)$$

Now, from the solutions (8)–(29) we have the following types of solutions:

Type 1. Substituting $s = \frac{3h^2}{16p}$, $r = \frac{16p^2}{27h}$, into the algebraic Eqs. (260) and solve them by Maple, one gets the following results:

$$\begin{aligned}
&\delta_0 = \frac{32\Delta_2}{3\left(11 + 3\sqrt{33}\right)[b_1 - \kappa(\lambda+\nu)]}, \quad \delta_1 = 0,\\
&\delta_2 = -\frac{4h\Delta_0}{3\left(1 + \sqrt{33}\right)[b_1 - \kappa(\lambda+\nu)]},\\
&p = -\frac{3\left(11 + \sqrt{33}\right)\Delta_2}{11\Delta_0}, \quad h = h,\\
&b_2 = \frac{9\left(77 - 3\sqrt{33}\right)[b_1 - \kappa(\lambda+\nu)]^2}{1024\Delta_2},\\
&b_3 = \frac{1280\Delta_2^3}{33\left(33 + \sqrt{33}\right)[b_1 - \kappa(\lambda+\nu)]^2}, \quad \alpha = \frac{3}{2}\Delta_0,
\end{aligned} \quad (261)$$

provided $b_1 - \kappa(\lambda+\nu) \neq 0$ and $\Delta_2 \neq 0$. If we substitute (261) along with (8)–(12) into Eq. (90), then Eq. (256) has the following solutions.

11.1.1. Soliton solutions.

$$q(x,t)=\left\{\frac{32\Delta_2}{3\left(11+3\sqrt{33}\right)[b_1-\kappa(\lambda+\nu)]}\left[1-\frac{8\left(5+\sqrt{33}\right)\tanh^2\left(\epsilon\sqrt{\frac{\left(11+\sqrt{33}\right)\Delta_2}{11\Delta_0}}(x-ct)\right)}{\left(1+\sqrt{33}\right)\left[3+\tanh^2\left(\epsilon\sqrt{\frac{\left(11+\sqrt{33}\right)\Delta_2}{11\Delta_0}}(x-ct)\right)\right]}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \quad (262)$$

and

$$q(x,t)=\left\{\frac{32\Delta_2}{3\left(11+3\sqrt{33}\right)[b_1-\kappa(\lambda+\nu)]}\left[1-\frac{8\left(5+\sqrt{33}\right)\coth^2\left(\epsilon\sqrt{\frac{\left(11+\sqrt{33}\right)\Delta_2}{11\Delta_0}}(x-ct)\right)}{\left(1+\sqrt{33}\right)\left[3+\coth^2\left(\epsilon\sqrt{\frac{\left(11+\sqrt{33}\right)\Delta_2}{11\Delta_0}}(x-ct)\right)\right]}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \quad (263)$$

provided $\Delta_2[b_1-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2>0$ and $\epsilon=\pm1$.

11.1.2. Periodic solutions.

$$q(x,t)=\left\{\frac{32\Delta_2}{3\left(11+3\sqrt{33}\right)[b_1-\kappa(\lambda+\nu)]}\left[1+\frac{8\left(5+\sqrt{33}\right)\tan^2\left(\epsilon\sqrt{-\frac{\left(11+\sqrt{33}\right)\Delta_2}{11\Delta_0}}(x-ct)\right)}{\left(1+\sqrt{33}\right)\left[3-\tan^2\left(\epsilon\sqrt{-\frac{\left(11+\sqrt{33}\right)\Delta_2}{11\Delta_0}}(x-ct)\right)\right]}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \quad (264)$$

and

$$q(x,t)=\left\{\frac{32\Delta_2}{3\left(11+3\sqrt{33}\right)[b_1-\kappa(\lambda+\nu)]}\left[1+\frac{8\left(5+\sqrt{33}\right)\cot^2\left(\epsilon\sqrt{-\frac{\left(11+\sqrt{33}\right)\Delta_2}{11\Delta_0}}(x-ct)\right)}{\left(1+\sqrt{33}\right)\left[3-\cot^2\left(\epsilon\sqrt{-\frac{\left(11+\sqrt{33}\right)\Delta_2}{11\Delta_0}}(x-ct)\right)\right]}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \quad (265)$$

provided $\Delta_2[b_1-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

Type 2. Substituting $s=\frac{3h^2}{16p}$, $r=0$, into the algebraic Eqs. (260) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=\epsilon\sqrt{\frac{2b_3}{5\Delta_2}},\quad \delta_1=0,\quad \delta_2=\frac{\epsilon h\Delta_0}{8\Delta_2}\sqrt{\frac{2b_3}{5\Delta_2}},\\ p=\frac{4\Delta_2}{\Delta_0},\quad h=h,\quad b_2=-\frac{15\Delta_2^2}{4b_3},\\ \alpha=\frac{3}{2}\Delta_0,\quad b_1=\kappa(\lambda+\nu),\end{gathered} \quad (266)$$

provided $b_3\Delta_2>0$, $\Delta_0\neq0$ and $\epsilon=\pm1$.

If we substitute (266) along with (14) and (15) into Eq. (90), then Eq. (256) has the following solutions.

11.1.3. Dark and singular solutions.

$$q(x,t)=\left\{\sqrt{\frac{2b_3}{5\Delta_2}}\tanh\left(2\epsilon\sqrt{\frac{\Delta_2}{\Delta_0}}(x-ct)\right)\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \quad (267)$$

and

$$q(x,t)=\left\{\sqrt{\frac{2b_3}{5\Delta_2}}\coth\left(2\epsilon\sqrt{\frac{\Delta_2}{\Delta_0}}(x-ct)\right)\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \quad (268)$$

respectively, provided $\Delta_2b_3>0$, $\Delta_0\Delta_2>0$ and $\epsilon=\pm1$.

Type 3. Substituting $r=0$, into the algebraic Eqs. (260) and solve them by Maple, we get the following results:

$$\delta_0 = -\frac{10\Delta_2}{7[b_1 - \kappa(\lambda+\nu)]}, \quad \delta_1 = 0, \quad \delta_2 = -\frac{20\Delta_2}{7[b_1 - \kappa(\lambda+\nu)]}, \quad p = -\frac{2\Delta_2}{7\Delta_0},$$
$$s = -\frac{96\Delta_2}{7\Delta_0}, \quad h = -\frac{48\Delta_2}{7\Delta_0}, \quad b_2 = \frac{21[b_1 - \kappa(\lambda+\nu)]^2}{25\Delta_2}, \tag{269}$$
$$b_3 = \frac{500\Delta_2^3}{343[b_1 - \kappa(\lambda+\nu)]^2}, \quad \alpha = \frac{3}{2}\Delta_0,$$

provided $b_1 - \kappa(\lambda+\nu) \neq 0$, $\Delta_0 \neq 0$ and $\Delta_2 \neq 0$.

If we substitute (269) along with (16)–(29) into Eq. (90), then Eq. (256) has the following solutions.

11.1.4. Soliton solutions.

$$q(x,t) = \left\{-\frac{10\Delta_2}{7[b_1 - \kappa(\lambda+\nu)]}\left[1 - \frac{3\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}{18 - 2\left[1 + \tanh\left(\epsilon\sqrt{-\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{270}$$

$$q(x,t) = \left\{-\frac{10\Delta_2}{7[b - \kappa(\lambda+\nu)]}\left[1 + \frac{12\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}{18 - 2\left[1 + \coth\left(\epsilon\sqrt{-\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{271}$$

$$q(x,t) = \left\{-\frac{10\Delta_2}{7[b_1 - \kappa(\lambda+\nu)]}\left[1 - \frac{\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}{6 + 4\tanh\left(\epsilon\sqrt{-\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{272}$$

$$q(x,t) = \left\{-\frac{10\Delta_2}{7[b_1 - \kappa(\lambda+\nu)]}\left[1 + \frac{3\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}{6 + 4\coth\left(\epsilon\sqrt{-\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{273}$$

provided $\Delta_2[b_1 - \kappa(\lambda+\nu)] < 0$, $\Delta_0\Delta_2 < 0$ and $\epsilon = \pm 1$.

11.1.5. Bright soliton.

$$q(x,t) = \left\{-\frac{10\Delta_2}{7[b_1 - \kappa(\lambda+\nu)]}\left[1 - \frac{1}{\sqrt{5}\cosh\left(2\epsilon\sqrt{-\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right) + 3}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{274}$$

and

$$q(x,t) = \left\{-\frac{10\Delta_2}{7[b_1 - \kappa(\lambda+\nu)]}\left[1 - \frac{1}{2\sqrt{5}\cosh^2\left(\epsilon\sqrt{-\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right) - \left(\sqrt{5}-3\right)}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{275}$$

provided $\Delta_2[b_1 - \kappa(\lambda+\nu)] < 0$, $\Delta_0\Delta_2 < 0$ and $\epsilon = \pm 1$.

11.1.6. Singular soliton.

$$q(x,t)=\left\{-\frac{10\Delta_2}{7[b_1-\kappa(\lambda+\nu)]}\left[1-\frac{1}{2\sqrt{5}\sinh^2\left(\epsilon\sqrt{-\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)+\left(\sqrt{5}-3\right)}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{276}$$

provided $\Delta_2[b_1-\kappa(\lambda+\nu)]<0, \Delta_0\Delta_2<0$ and $\epsilon=\pm 1$.

11.1.7. Periodic solutions.

$$q(x,t)=\left\{-\frac{10\Delta_2}{7[b_1-\kappa(\lambda+\nu)]}\left[1-\frac{\sec\left(2\epsilon\sqrt{\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}{\sqrt{5}+3\sec\left(2\epsilon\sqrt{\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{277}$$

$$q(x,t)=\left\{-\frac{10\Delta_2}{7[b_1-\kappa(\lambda+\nu)]}\left[1-\frac{\operatorname{cosec}\left(2\epsilon\sqrt{\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}{\sqrt{5}+3\operatorname{cosec}\left(2\epsilon\sqrt{\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{278}$$

$$q(x,t)=\left\{-\frac{10\Delta_2}{7[b_1-\kappa(\lambda+\nu)]}\left[1-\frac{\sec^2\left(\epsilon\sqrt{\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}{2\sqrt{5}-\left(\sqrt{5}-3\right)\sec^2\left(\epsilon\sqrt{\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{279}$$

$$q(x,t)=\left\{-\frac{10\Delta_2}{7[b_1-\kappa(\lambda+\nu)]}\left[1-\frac{\operatorname{cosec}^2\left(\epsilon\sqrt{\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}{2\sqrt{5}-\left(\sqrt{5}-3\right)\operatorname{cosec}^2\left(\epsilon\sqrt{\frac{2\Delta_2}{7\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{280}$$

provided $\Delta_2[b_1-\kappa(\lambda+\nu)]<0$, $\Delta_0\Delta_2>0$ and $\epsilon=\pm 1$.

11.2. Addendum to Kudryashov's Method

According to this method, we balance UU'' with U^4 in Eq. (259), one gets the same relation (146). Now, we will discuss the following cases:

Case 1. If we choose $p=1$, then $N=1$. Thus, we deduce that from (31) that Eq. (259) has the same formal solution (113). Substituting (113) along with (32) into Eq. (259), collecting all the coefficients of each power of $[R(\xi)]^{m_1}[R'(\xi)]^j$, $(m_1=0,1,2,\ldots,4,\ j=0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
-4b_2\beta_1^4+3\Delta_0\beta_1^2\chi\ln^2k-\beta_1^2\alpha\chi\ln^2k=0,\\
4\Delta_0\beta_1\beta_0\chi\ln^2k-4[b_1-\kappa(\lambda+\nu)]\beta_1^3-16b_2\beta_0\beta_1^3=0,\\
\beta_1^2\alpha\ln^2k+4\Delta_2\beta_1^2-12[b_1-\kappa(\lambda+\nu)]\beta_0\beta_1^2-24b_2\beta_0^2\beta_1^2-\Delta_0\beta_1^2\ln^2k=0,\\
-2\Delta_0\beta_1\beta_0\ln^2k+8\Delta_2\beta_0\beta_1-12[b_1-\kappa(\lambda+\nu)]\beta_0^2\beta_1-16b_2\beta_0^3\beta_1=0,\\
-4b_3+4\Delta_2\beta_0^2-4[b_1-\kappa(\lambda+\nu)]\beta_0^3-4b_2\beta_0^4=0.
\end{gathered} \tag{281}$$

On solving the above algebraic Eqs. (281) by using the Maple, one gets the following results:

$$\beta_0 = \frac{2\Delta_2 - \Delta_0 \ln^2 k}{2[b_1 - \kappa(\lambda+\nu)]}, \quad \beta_1 = \frac{\epsilon\sqrt{2\chi}\left(\Delta_0 \ln^2 k - 2\Delta_2\right)}{2[b_1 - \kappa(\lambda+\nu)]}, \quad \alpha = \frac{2\Delta_2 + \Delta_0 \ln^2 k}{\ln^2 k},$$
$$b_2 = \frac{\left(\Delta_0 \ln^2 k - \Delta_2\right)[b_1 - \kappa(\lambda+\nu)]^2}{\left(\Delta_0 \ln^2 k - 2\Delta_2\right)^2}, \quad b_3 = \frac{\left(\Delta_0 \ln^2 k + \Delta_2\right)\left(\Delta_0 \ln^2 k - 2\Delta_2\right)^2}{16[b_1 - \kappa(\lambda+\nu)]^2}, \tag{282}$$

provided $\chi > 0$ and $\epsilon = \pm 1$.

Substituting (282) along with (33) into Eq. (113), one gets the solutions of Eq. (256) in the form:

$$q(x,t) = \left\{\frac{2\Delta_2 - \Delta_0 \ln^2 k}{2[b_1 - \kappa(\lambda+\nu)]}\left[1 + \frac{4\epsilon A\sqrt{2\chi}}{4A^2 \exp_k[(x-ct)] + \chi \exp_k[-(x-ct)]}\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{283}$$

provided $\left(2\Delta_2 - \Delta_0 \ln^2 k\right)[b_1 - \kappa(\lambda+\nu)] > 0$ and $\epsilon = \pm 1$.

In particular, if we set $\chi = 4A^2$ in (283), then we have the bright soliton solution of Eq. (256) as:

$$q(x,t) = \left\{\frac{2\Delta_2 - \Delta_0 \ln^2 k}{2[b_1 - \kappa(\lambda+\nu)]}\left[1 + \epsilon\sqrt{2}\operatorname{sech}[(x-ct)\ln k]\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta_0)}. \tag{284}$$

Case 2. If we choose $p = 2$, then $N = 2$. Thus, we deduce that from (31) that Eq. (259) has the same formal solution (119). Substituting (119) along with Eq. (32) into Eq. (259), collecting all the coefficients of each power of $[R(\xi)]^{m_2}$ $[R'(\xi)]^j$, $(m_2 = 0,1,2,\ldots,8,\ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
-4b_2\beta_2^4 - 4\alpha\chi\beta_2^2 \ln^2 k + 12\Delta_0\beta_2^2\chi \ln^2 k = 0,\\
18\Delta_0\beta_1\beta_2\chi \ln^2 k - 4\alpha\chi\beta_1\beta_2 \ln^2 k - 16b_2\beta_1\beta_2^3 = 0,\\
-16b_2\beta_0\beta_2^3 - \beta_1^2\alpha\chi \ln^2 k + 16\Delta_0\beta_0\beta_2\chi \ln^2 k - 4[b_1 - \kappa(\lambda+\nu)]\beta_2^3 - 24b_2\beta_1^2\beta_2^2 + 5\Delta_0\beta_1^2 \ln^2 k\chi = 0,\\
-12[b_1 - \kappa(\lambda+\nu)]\beta_1\beta_2^2 - 48b_2\beta_0\beta_1\beta_2^2 - 16b_2\beta_1^3\beta_2 + 6\Delta_0\beta_1\beta_0\chi \ln^2 k = 0,\\
-48b_2\beta_0\beta_1^2\beta_2 - 4\Delta_0\beta_2^2 \ln^2 k + 4\alpha\beta_2^2 \ln^2 k + 4\Delta_2\beta_2^2 - 24b_2\beta_0^2\beta_2^2 - 4b_2\beta_1^4\\
- 12[b_1 - \kappa(\lambda+\nu)]\beta_1^2\beta_2 - 12[b_1 - \kappa(\lambda+\nu)]\beta_0\beta_2^2 = 0,\\
-6\Delta_0\beta_1\beta_2 \ln^2 k - 24[b_1 - \kappa(\lambda+\nu)]\beta_0\beta_1\beta_2 + 4\alpha\beta_1\beta_2 \ln^2 k - 16b_2\beta_0\beta_1^3\\
- 4[b_1 - \kappa(\lambda+\nu)]\beta_1^3 - 48b_2\beta_0^2\beta_1\beta_2 + 8\Delta_2\beta_1\beta_2 = 0,\\
-12[b_1 - \kappa(\lambda+\nu)]\beta_0^2\beta_2 + \beta_1^2\alpha \ln^2 k + 4\Delta_2\beta_1^2 - \Delta_0\beta_1^2 \ln^2 k - 8\Delta_0\beta_0\beta_2 \ln^2 k\\
+8\Delta_2\beta_0\beta_2 - 24b_2\beta_0^2\beta_1^2 - 16b_2\beta_0^3\beta_2 - 12[b_1 - \kappa(\lambda+\nu)]\beta_0\beta_1^2 = 0,\\
-2\Delta_0\beta_1\beta_0 \ln^2 k + 8\Delta_2\beta_0\beta_1 - 12[b_1 - \kappa(\lambda+\nu)]\beta_0^2\beta_1 - 16b_2\beta_0^3\beta_1 = 0,\\
-4[b_1 - \kappa(\lambda+\nu)]\beta_0^3 - 4b_3 - 4b_2\beta_0^4 + 4\Delta_2\beta_0^2 = 0.
\end{gathered} \tag{285}$$

On solving the above algebraic Eqs. (285) by using the Maple, one gets the following results:

$$\beta_0 = \frac{\Delta_2 - 2\Delta_0 \ln^2 k}{b_1 - \kappa(\lambda+\nu)}, \quad \beta_1 = 0, \quad \beta_2 = \frac{\epsilon\sqrt{2\chi}\left(2\Delta_0 \ln^2 k - \Delta_2\right)}{b_1 - \kappa(\lambda+\nu)}, \quad \alpha = \frac{\Delta_2 + 2\Delta_0 \ln^2 k}{2\ln^2 k},$$
$$b_2 = \frac{\left(4\Delta_0 \ln^2 k - \Delta_2\right)[b_1 - \kappa(\lambda+\nu)]^2}{4\left(2\Delta_0 \ln^2 k - \Delta_2\right)^2}, \quad b_3 = \frac{16\Delta_0^3 \ln^6 k - 12\Delta_0^2\Delta_2 \ln^4 k + \Delta_2^3}{4[b_1 - \kappa(\lambda+\nu)]^2}, \tag{286}$$

provided $\chi > 0$ and $\epsilon = \pm 1$.

Substituting (286) along with (33) into Eq. (119), one gets the solutions of Eq. (256) in the form:

$$q(x,t)=\left\{\frac{\Delta_2-2\Delta_0\ln^2 k}{b_1-\kappa(\lambda+\nu)}\left[1+\frac{4\epsilon A\sqrt{2\chi}}{4A^2\exp_k\left[2(x-ct)\right]+\chi\exp_k\left[-2(x-ct)\right]}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\quad(287)$$

provided $\left(\Delta_2-2\Delta_0\ln^2 k\right)\left[b_1-\kappa(\lambda+\nu)\right]>0$.

In particular, if we set $\chi=4A^2$ in (287), then we have the bright soliton solution of Eq. (256) as:

$$q(x,t)=\left\{\frac{\Delta_2-2\Delta_0\ln^2 k}{b_1-\kappa(\lambda+\nu)}\left[1+\epsilon\sqrt{2}\operatorname{sech}\left[2(x-ct)\ln k\right]\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)}.\quad(288)$$

Similarly, we can find many other solutions by choosing other values for p and N.

12. GENERALIZED ANTI-CUBIC LAW

For the generalized anti-cubic law nonlinearity, we have

$$F(\phi)=b_1\phi^n+b_2\phi^{n+1}+\frac{b_3}{\phi^{n+1}},\quad(289)$$

where b_1, b_2 and b_3 are constants. Equation (1) corresponding to generalized anti-cubic law nonlinearity (289) is given by:

$$\begin{aligned}&iq_t+iaq_{xxx}+\left(b_1|q|^{2n}+b_2|q|^{2(n+1)}+\frac{b_3}{|q|^{2(n+1)}}\right)q=\alpha\frac{|q_x|^2}{q^*}\\&+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q\\&+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_xq+\nu|q|^{2m}q_x\right],\end{aligned}\quad(290)$$

where Eq. (41) reduces to:

$$\begin{aligned}&\Delta_0\phi\phi''-\alpha\phi'^2-\Delta_2\phi-\kappa(\lambda+\nu)\phi^{2m+2}\\&+b_1\phi^{2n+2}+b_2\phi^{2n+4}+b_3\phi^{-2n}=0.\end{aligned}\quad(291)$$

For integrability, one must select $n=m$. This leads to the modification of Eq. (1) corresponding to anti-cubic law nonlinearity as:

$$\begin{aligned}&iq_t+iaq_{xxx}+\left(b_1|q|^{2m}+b_2|q|^{2(m+1)}+\frac{b_3}{|q|^{2(m+1)}}\right)q\\&=\alpha\frac{|q_x|^2}{q^*}+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q\\&+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_xq+\nu|q|^{2m}q_x\right].\end{aligned}\quad(292)$$

Consequently, Eq. (291) becomes:

$$\begin{aligned}&\Delta_0\phi^{2m+1}\phi''-\alpha\phi^{2m}\phi'^2-\Delta_2\phi^{2m+2}\\&+\left[b_1-\kappa(\lambda+\nu)\right]\phi^{4m+2}+b_2\phi^{4m+4}+b_3=0.\end{aligned}\quad(293)$$

Balancing $\phi^{2m+1}\phi''$ and ϕ^{4m+4} in Eq. (293), gives the balance number $N=\frac{1}{m+1}$. Since the balance number is not integer, then we take into consideration the transformation

$$\phi(\xi)=\left[U(\xi)\right]^{\frac{1}{m+1}},\quad(294)$$

where $U(\xi)$ is a new function of ξ. Substituting (294) into (293), we have the new equation

$$\begin{aligned}&(m+1)\Delta_0UU''-(m\Delta_0+\alpha)U'^2\\&+(m+1)^2\left[b_1-\kappa(\lambda+\nu)\right]U^{\frac{4m+2}{m+1}}\\&+(m+1)^2\left[b_3-\Delta_2U^2+b_2U^4\right]=0.\end{aligned}\quad(295)$$

For integrability, one must select

$$b_1=\kappa(\lambda+\nu).\quad(296)$$

This leads to the modification of Eq. (1) corresponding to generalized anti-cubic law nonlinearity as:

$$\begin{aligned}&iq_t+iaq_{xxx}+\left(\kappa(\lambda+\nu)|q|^{2m}+b_2|q|^{2(m+1)}+\frac{b_3}{|q|^{2(m+1)}}\right)q\\&=\alpha\frac{|q_x|^2}{q^*}+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]\\&+\gamma q+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_xq+\nu|q|^{2m}q_x\right].\end{aligned}\quad(297)$$

Consequently, Eq. (295) changes to:

$$\begin{aligned}&(m+1)\Delta_0UU''-(m\Delta_0+\alpha)U'^2\\&+(m+1)^2\left[b_3-\Delta_2U^2+b_2U^4\right]=0.\end{aligned}\quad(298)$$

In the next two subsections, we will solve Eq. (298) using the following two methods.

12.1. New Mapping Method

According to the new mapping method, we balance UU'' with U^4 in Eq. (298) yields the balance number $N=1$. Now, from (6), the solution of Eq. (298) has the same formal solution (90). Substituting (90) along with (7) into Eq. (298), collecting all the coefficients of $F^l(\xi)[F'(\xi)]^j$ $(l = 0,1,\ldots,8,\ j = 0,1)$ and setting them to zero, we have the following algebraic equations:

$$\begin{gathered}
\frac{4}{3}\Delta_0\delta_2^2 ms + m^2 b_2\delta_2^4 - \frac{4}{3}\alpha s\delta_2^2 \\
+ b_2\delta_2^4 + 2mb_2\delta_2^4 + \frac{8}{3}\Delta_0\delta_2^2 s = 0, \\
4b_2\delta_1\delta_2^3 + 8mb_2\delta_1\delta_2^3 + \frac{7}{3}\Delta_0\delta_1 m\delta_2 s - \frac{4}{3}\alpha s\delta_1\delta_2 \\
+ \frac{11}{3}\Delta_0\delta_1\delta_2 s + 4m^2 b_2\delta_1\delta_2^3 = 0, \\
4m^2 b_2\delta_0\delta_2^3 - \frac{1}{3}\alpha s\delta_1^2 + 3\Delta_0\delta_2^2 h \\
+ \frac{8}{3}\Delta_0\delta_0 m\delta_2 s + 8mb_2\delta_0\delta_2^3 + \frac{2}{3}\Delta_0\delta_1^2 ms + 6m^2 b_2\delta_1^2\delta_2^2 \\
+ 6b_2\delta_1^2\delta_2^2 + \frac{8}{3}\Delta_0\delta_0\delta_2 s + 12mb_2\delta_1^2\delta_2^2 - 2\alpha h\delta_2^2 \\
+ \Delta_0\delta_1^2 s + 4b_2\delta_0\delta_2^3 + \Delta_0\delta_2^2 mh = 0, \\
-2\alpha h\delta_1\delta_2 + 2\Delta_0\delta_1 m\delta_2 h + 12m^2 b_2\delta_0\delta_1\delta_2^2 \\
+ \Delta_0\delta_0 m\delta_1 s + 4\Delta_0\delta_1\delta_2 h + 12b_2\delta_0\delta_1\delta_2^2 \\
+ 4m^2 b_2\delta_1^3\delta_2 + 4b_2\delta_1^3\delta_2 + 8mb_2\delta_1^3\delta_2 \\
+ 24mb_2\delta_0\delta_1\delta_2^2 + \Delta_0\delta_0\delta_1 s = 0, \\
4\Delta_0\delta_2^2 p + 3\Delta_0\delta_0\delta_2 h + \Delta_0\delta_1^2 h + 12m^2 b_2\delta_0\delta_1^2\delta_2 \\
- m^2\Delta_2\delta_2^2 + 6b_2\delta_0^2\delta_2^2 - \Delta_2\delta_2^2 \\
+ 3\Delta_0\delta_0 m\delta_2 h + 12b_2\delta_0\delta_1^2\delta_2 + \frac{1}{2}\Delta_0\delta_1^2 mh \\
+ m^2 b_2\delta_1^4 - 2m\Delta_2\delta_2^2 + 12mb_2\delta_0^2\delta_2^2 \\
- \frac{1}{2}\alpha h\delta_1^2 + b_2\delta_1^4 + 24mb_2\delta_0\delta_1^2\delta_2 \\
+ 6m^2 b_2\delta_0^2\delta_2^2 - 4\alpha p\delta_2^2 + 2mb_2\delta_1^4 = 0, \\
5\Delta_0\delta_1\delta_2 p + \Delta_0\delta_1 m\delta_2 p + 12m^2 b_2\delta_0^2\delta_1\delta_2 \\
- 4m\Delta_2\delta_1\delta_2 - 2\Delta_2\delta_1\delta_2 + 8mb_2\delta_0\delta_1^3 \\
- 2m^2\Delta_2\delta_1\delta_2 + 24mb_2\delta_0^2\delta_1\delta_2 + \Delta_0\delta_0 m\delta_1 h \\
+ 12b_2\delta_0^2\delta_1\delta_2 + 4m^2 b_2\delta_0\delta_1^3 \\
+ \Delta_0\delta_0\delta_1 h + 4b_2\delta_0\delta_1^3 - 4\alpha p\delta_1\delta_2 = 0, \\
8mb_2\delta_0^3\delta_2 + 12mb_2\delta_0^2\delta_1^2 + 4m^2 b_2\delta_0^3\delta_2 \\
- \Delta_2\delta_1^2 + 6m^2 b_2\delta_0^2\delta_1^2 - 2\Delta_2\delta_0\delta_2 \\
- 2\Delta_0\delta_2^2 mr + 4\Delta_0\delta_0\delta_2 p - m^2\Delta_2\delta_1^2 \\
- 4m\Delta_2\delta_0\delta_2 + 2\Delta_0\delta_2^2 r + \Delta_0\delta_1^2 p - \alpha p\delta_1^2 \\
+ 4\Delta_0\delta_0 m\delta_2 p + 4b_2\delta_0^3\delta_2 + 6b_2\delta_0^2\delta_1^2 \\
- 4\alpha r\delta_2^2 - 2m^2\Delta_2\delta_0\delta_2 - 2m\Delta_2\delta_1^2 = 0, \\
-2\Delta_0\delta_1 m\delta_2 r - 2m^2\Delta_2\delta_0\delta_1 - 4m\Delta_2\delta_0\delta_1 \\
- 2\Delta_2\delta_0\delta_1 + 8mb_2\delta_0^3\delta_1 + 2\Delta_0\delta_1\delta_2 r \\
+ \Delta_0\delta_0\delta_1 p + 4b_2\delta_0^3\delta_1 + 4m^2 b_2\delta_0^3\delta_1 \\
+ \Delta_0\delta_0 m\delta_1 p - 4\alpha r\delta_1\delta_2 = 0, \\
2b_3 m + b_3 m^2 + b_3 + 2mb_2\delta_0^4 - m^2\Delta_2\delta_0^2 \\
+ m^2 b_2\delta_0^4 + 2\Delta_0\delta_0\delta_2 r - \Delta_0 mr\delta_1^2 \\
-\Delta_2\delta_0^2 + b_2\delta_0^4 - \alpha r\delta_1^2 - 2m\Delta_2\delta_0^2 + 2\Delta_0\delta_0 m\delta_2 r = 0.
\end{gathered} \tag{299}$$

12.1.1. Soliton solutions.

With the aid of the solutions (8)–(29), we have the following types of solutions:

Type 1. Substituting $s = \dfrac{3h^2}{16p}$, $r = \dfrac{16p^2}{27h}$, into the algebraic Eqs. (299) and solve them by Maple, one gets the following results:

$$\begin{gathered}
\delta_0 = -\frac{\epsilon}{3\left(11+\sqrt{33}\right)}\sqrt{\frac{22b_3\left(33-\sqrt{33}\right)}{\Delta_2}}, \quad \delta_1 = 0, \\
\delta_2 = \frac{\epsilon h\Delta_0}{24(m+1)\Delta_2}\sqrt{\frac{22b_3\left(33-\sqrt{33}\right)}{\Delta_2}}, \\
p = -\frac{6(m+1)\Delta_2}{\left(11-\sqrt{33}\right)\Delta_0}, \quad h = h, \\
b_2 = \frac{6\left(11-\sqrt{33}\right)\Delta_2^2}{11\left(33-\sqrt{33}\right)b_3}, \quad \alpha = \frac{1}{2}(m+3)\Delta_0,
\end{gathered} \tag{300}$$

provided $b_3\Delta_2 > 0$, $\Delta_0 \neq 0$ and $\epsilon = \pm 1$.

If we substitute (300) along with (8)–(12) into Eq. (90), then Eq. (297) has the following solutions.

$$q(x,t) = \left\{\frac{1}{3\left(11+\sqrt{33}\right)}\sqrt{\frac{22b_3\left(33-\sqrt{33}\right)}{\Delta_2}}\left[\frac{-3+\left(6+\sqrt{33}\right)\tanh^2\left(\epsilon\sqrt{\frac{2(1+m)\Delta_2}{\left(11-\sqrt{33}\right)\Delta_0}}(x-ct)\right)}{3+\tanh^2\left(\epsilon\sqrt{\frac{2(1+m)\Delta_2}{\left(11-\sqrt{33}\right)\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m+1}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{301}$$

and

$$q(x,t)=\left\{\frac{1}{3\left(11+\sqrt{33}\right)}\sqrt{\frac{22b_3\left(33-\sqrt{33}\right)}{\Delta_2}}\left[\frac{-3+\left(6+\sqrt{33}\right)\coth^2\left(\epsilon\sqrt{\frac{2(1+m)\Delta_2}{\left(11-\sqrt{33}\right)\Delta_0}}(x-ct)\right)}{3+\coth^2\left(\epsilon\sqrt{\frac{2(1+m)\Delta_2}{\left(11-\sqrt{33}\right)\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m+1}} e^{i(-\kappa x+\omega t+\theta_0)}, \quad (302)$$

provided $\Delta_0\Delta_2>0, b_3\Delta_2>0$ and $\varepsilon=\pm1$.

12.1.2. Periodic solutions.

$$q(x,t)=\left\{\frac{1}{3\left(11+\sqrt{33}\right)}\sqrt{\frac{22b_3\left(33-\sqrt{33}\right)}{\Delta_2}}\left[\frac{3+\left(6+\sqrt{33}\right)\tan^2\left(\epsilon\sqrt{-\frac{2(1+m)\Delta_2}{\left(11-\sqrt{33}\right)\Delta_0}}(x-ct)\right)}{3-\tan^2\left(\epsilon\sqrt{-\frac{2(1+m)\Delta_2}{\left(11-\sqrt{33}\right)\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m+1}} e^{i(-\kappa x+\omega t+\theta_0)}. \quad (303)$$

and

$$q(x,t)=\left\{\frac{1}{3\left(11+\sqrt{33}\right)}\sqrt{\frac{22b_3\left(33-\sqrt{33}\right)}{\Delta_2}}\left[\frac{3+\left(6+\sqrt{33}\right)\cot^2\left(\epsilon\sqrt{-\frac{2(1+m)\Delta_2}{\left(11-\sqrt{33}\right)\Delta_0}}(x-ct)\right)}{3-\cot^2\left(\epsilon\sqrt{-\frac{2(1+m)\Delta_2}{\left(11-\sqrt{33}\right)\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m+1}} e^{i(-\kappa x+\omega t+\theta_0)}, \quad (304)$$

provided $\Delta_0\Delta_2<0, b_3\Delta_2>0$ and $\epsilon=\pm1$.

Type 2. Substituting $s=\frac{3h^2}{16p}$, $r=0$, into the algebraic Eqs. (300) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=\epsilon\sqrt{\frac{2mb_3}{(3m+2)\Delta_2}},\quad \delta_1=0,\\ \delta_2=\frac{\epsilon mh\Delta_0}{(m+1)^2\Delta_2}\sqrt{\frac{2mb_3}{(3m+2)\Delta_2}},\\ p=\frac{(m+1)^2\Delta_2}{m\Delta_0},\quad h=h,\\ b_2=-\frac{(m+2)(3m+2)\Delta_2^2}{4m^2b_3},\quad \alpha=\left(m+\frac{1}{2}\right)\Delta_0,\end{gathered} \quad (305)$$

provided $b_3\Delta_2>0$, $\Delta_0\neq0$ and $\epsilon=\pm1$.

If we substitute (305) along with (14) and (15) into Eq. (90), then Eq. (297) has the following solutions.

12.1.3. Dark and singular solitons.

$$\begin{aligned}q(x,t)&=\left\{\sqrt{\frac{2mb_3}{(3m+2)\Delta_2}}\right.\\ &\times\left.\tanh\left(\varepsilon(m+1)\sqrt{\frac{\Delta_2}{m\Delta_0}}(x-ct)\right)\right\}^{\frac{1}{m+1}} e^{i(-\kappa x+\omega t+\theta_0)},\end{aligned} \quad (306)$$

and

$$\begin{aligned}q(x,t)&=\left\{\sqrt{\frac{2mb_3}{(3m+2)\Delta_2}}\right.\\ &\times\left.\coth\left(\varepsilon(m+1)\sqrt{\frac{\Delta_2}{m\Delta_0}}(x-ct)\right)\right\}^{\frac{1}{m+1}} e^{i(-\kappa x+\omega t+\theta_0)},\end{aligned} \quad (307)$$

respectively, provided $\Delta_2 b_3>0$, $\Delta_0\Delta_2>0$ and $\epsilon=\pm1$.

Type 3. Substituting $r=0$, into the algebraic Eqs. (300) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=\sqrt{\frac{7b_3}{3\Delta_2}},\quad \delta_1=0,\quad \delta_2=2\sqrt{\frac{7b_3}{3\Delta_2}},\\ p=-\frac{(m+1)\Delta_2}{14\Delta_0},\quad s=-\frac{24(m+1)\Delta_2}{7\Delta_0},\\ h=-\frac{12(m+1)\Delta_2}{7\Delta_0},\quad b_2=\frac{12\Delta_2^2}{49b_3},\\ \alpha=\frac{1}{2}(m+3)\Delta_0,\end{gathered} \quad (308)$$

provided $b_3\Delta_2>0$, $\Delta_0\neq0$.

If we substitute (308) along with (16)–(29) into Eq. (90), then Eq. (297) has the following solutions.

12.1.4. Soliton solutions.

$$q(x,t)=\left\{\sqrt{\frac{7b_3}{3\Delta_2}}\left[1-\frac{3\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}{18-2\left[1+\tanh\left(\epsilon\sqrt{-\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{m+1}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{309}$$

$$q(x,t)=\left\{\sqrt{\frac{7b_3}{3\Delta_2}}\left[1+\frac{3\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}{18-2\left[1+\coth\left(\epsilon\sqrt{-\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{m+1}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{310}$$

$$q(x,t)=\left\{\sqrt{\frac{7b_3}{3\Delta_2}}\left[1-\frac{\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}{6-4\tanh\left(\epsilon\sqrt{-\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m+1}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{311}$$

$$q(x,t)=\left\{\sqrt{\frac{7b_3}{3\Delta_2}}\left[1+\frac{3\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}{6-4\coth\left(\epsilon\sqrt{-\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m+1}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{312}$$

provided $\Delta_2 b_3 > 0$, $\Delta_0\Delta_2 < 0$ and $\epsilon = \pm 1$.

12.1.5. Bright soliton.

$$q(x,t)=\left\{\sqrt{\frac{7b_3}{3\Delta_2}}\left[1-\frac{1}{\sqrt{5}\cosh\left(2\epsilon\sqrt{-\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)+3}\right]\right\}^{\frac{1}{m+1}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{313}$$

and

$$q(x,t)=\left\{\sqrt{\frac{7b_3}{3\Delta_2}}\left[1-\frac{1}{2\sqrt{5}\cosh^2\left(\epsilon\sqrt{-\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)-\left(\sqrt{5}-3\right)}\right]\right\}^{\frac{1}{m+1}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{314}$$

provided $\Delta_2 b_3 > 0$, $\Delta_0\Delta_2 < 0$ and $\epsilon = \pm 1$.

12.1.6. Singular soliton.

$$q(x,t)=\left\{\sqrt{\frac{7b_3}{3\Delta_2}}\left[1+\frac{1}{2\sqrt{5}\sinh^2\left(\epsilon\sqrt{-\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)+\left(\sqrt{5}-3\right)}\right]\right\}^{\frac{1}{m+1}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{315}$$

provided $\Delta_2 b_3 > 0$, $\Delta_0\Delta_2 < 0$ and $\epsilon = \pm 1$.

12.1.7. Periodic solutions.

$$q(x,t)=\left\{\sqrt{\frac{7b_3}{3\Delta_2}}\left[1-\frac{\sec\left(2\epsilon\sqrt{\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}{\sqrt{5}+3\sec\left(2\epsilon\sqrt{\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m+1}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{316}$$

$$q(x,t)=\left\{\sqrt{\frac{7b_3}{3\Delta_2}}\left[1-\frac{\operatorname{cosec}\left(2\epsilon\sqrt{\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}{\sqrt{5}+3\operatorname{cosec}\left(2\epsilon\sqrt{\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m+1}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{317}$$

$$q(x,t)=\left\{\sqrt{\frac{7b_3}{3\Delta_2}}\left[1-\frac{\sec^2\left(\epsilon\sqrt{\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}{2\sqrt{5}-\left(\sqrt{5}-3\right)\sec^2\left(\epsilon\sqrt{\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{318}$$

$$q(x,t)=\left\{\sqrt{\frac{7b_3}{3\Delta_2}}\left[1-\frac{\operatorname{cosec}^2\left(\epsilon\sqrt{\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}{2\sqrt{5}-\left(\sqrt{5}-3\right)\operatorname{cosec}^2\left(\epsilon\sqrt{\frac{(m+1)\Delta_2}{14\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{319}$$

provided $\Delta_2 b_3 > 0$, $\Delta_0\Delta_2 > 0$ and $\epsilon = \pm 1$.

12.2. Addendum to Kudryashov's Method

According to this method, we balance UU'' with U^4 in Eq. (298), one gets the same relation (146). Now, we will discuss the following cases:

Case 1. If we choose $p = 1$, then $N = 1$. Thus, we deduce that from (31) that Eq. (298) has the same formal solution (113). Substituting (113) along with (32) into Eq. (298), collecting all the coefficients of each power of $[R(\xi)]^{m_1}\,[R'(\xi)]^j$, $(m_1 = 0,1,2,\ldots,4,\ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
m^2b_2\beta_1^4-\Delta_0\beta_1^2m\chi\ln^2 k+2mb_2\beta_1^4+b_2\beta_1^4\\
+\beta_1^2\alpha\chi\ln^2 k-2\Delta_0\beta_1^2\chi\ln^2 k=0,\\
-2\Delta_0\beta_1\beta_0m\chi\ln^2 k+4b_2\beta_0\beta_1^3+8mb_2\beta_0\beta_1^3\\
-2\Delta_0\beta_1\beta_0\chi\ln^2 k+4m^2b_2\beta_0\beta_1^3=0,\\
-\Delta_2\beta_1^2+\Delta_0\beta_1^2\ln^2 k-\beta_1^2\alpha\ln^2 k\\
+6m^2b_2\beta_0^2\beta_1^2+12mb_2\beta_0^2\beta_1^2-m^2\Delta_2\beta_1^2\\
+6b_2\beta_0^2\beta_1^2-2m\Delta_2\beta_1^2=0,\\
-2m^2\Delta_2\beta_0\beta_1+\Delta_0\beta_1\beta_0m\ln^2 k+4b_2\beta_0^3\beta_1\\
+4m^2b_2\beta_0^3\beta_1+\Delta_0\beta_1\beta_0\ln^2 k\\
-2\Delta_2\beta_0\beta_1-4m\Delta_2\beta_0\beta_1+8mb_2\beta_0^3\beta_1=0,\\
+b_2\beta_0^4+2mb_3+m^2b_3+b_3+m^2b_2\beta_0^4\\
-2m\Delta_2\beta_0^2-\Delta_2\beta_0^2-m^2\Delta_2\beta_0^2+2mb_2\beta_0^4=0.
\end{gathered}\tag{320}$$

On solving the above algebraic Eqs. (320) by using the Maple, one gets the following results:

$$\begin{gathered}
\beta_0=\frac{2\epsilon}{\ln k}\sqrt{\frac{(1+m)b_3}{3\Delta_0}},\quad \beta_1=\frac{2\epsilon}{\ln k}\sqrt{\frac{2(1+m)\chi b_3}{3\Delta_0}},\\
\alpha=\frac{1}{2}(m+3)\Delta_0,\quad \Delta_2=\frac{\Delta_0\ln^2 k}{1+m},\quad b_2=\frac{3\Delta_0^2\ln^4 k}{16(1+m)^2b_3},
\end{gathered}\tag{321}$$

provided $\Delta_0 b_3 > 0$, $\chi > 0$ and $\epsilon = \pm 1$.

Substituting (321) along with (33) into Eq. (113), one gets the solutions of Eq. (297) in the form:

$$q(x,t)=\left\{\frac{2}{\ln k}\sqrt{\frac{(1+m)b_3}{3\Delta_0}}\left[1+\frac{4\epsilon A\sqrt{2\chi}}{4A^2\exp_k[(x-ct)]+\chi\exp_k[-(x-ct)]}\right]\right\}^{\frac{1}{m+1}}e^{i(-\kappa x+\omega t+\theta_0)}.\tag{322}$$

In particular, if we set $\chi = 4A^2$ in (322), then we have the bright soliton solution of Eq. (297) as:

$$q(x,t) = \left\{\frac{2}{\ln k}\sqrt{\frac{(1+m)b_3}{3\Delta_0}}\left[1+\epsilon\sqrt{2}\operatorname{sech}\left[(x-ct)\ln k\right]\right]\right\}^{\frac{1}{m+1}} e^{i(-\kappa x+\omega t+\theta_0)}. \tag{323}$$

Case 2. If we choose $p = 2$, then $N = 2$. Thus, we deduce that from (31) that Eq. (298) has the same formal solution (119). Substituting (119) along with Eq. (32) into Eq. (298), collecting all the coefficients of each power of $[R(\xi)]^{m_2}$ $[R'(\xi)]^j$, $(m_2 = 0,1,2,\ldots,8,\ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
-4\Delta_0 \ln^2 k\beta_2^2 m\chi - 8\Delta_0 \ln^2 k\beta_2^2\chi + b_2\beta_2^4 + m^2 b_2\beta_2^4 + 2mb_2\beta_2^4 + 4\ln^2 k\alpha\chi\beta_2^2 = 0,\\
4m^2 b_2\beta_1\beta_2^3 - 11\Delta_0 \ln^2 k\beta_1\beta_2\chi + 4b_2\beta_1\beta_2^3 + 8mb_2\beta_1\beta_2^2 + 4\ln^2 k\alpha\chi\beta_1\beta_2 - 7\Delta_0 \ln^2 k\beta_1 m\beta_2\chi = 0,\\
12mb_2\beta_1^2\beta_2^2 + 4b_2\beta_0\beta_2^3 + 6b_2\beta_1^2\beta_2^2 - 8\Delta_0 \ln^2 k\beta_0\beta_2\chi + 8mb_2\beta_0\beta_2^3 + \ln^2 k\alpha\chi\beta_1^2\\
- 2\Delta_0 \ln^2 k\beta_1^2 m\chi + 6m^2 b_2\beta_1^2\beta_2^2 - 3\Delta_0 \ln^2 k\beta_1 \wedge 2\chi + 4m \wedge 2b_2\beta_0\beta_2 \wedge 3 - 8\Delta_0 \ln^2 k\beta_0 m\beta_2\chi = 0,\\
8mb_2\beta_1^3\beta_2 - 3\Delta_0 \ln^2 k\beta_0\beta_1\chi + 4b_2\beta_1^3\beta_2 + 12b_2\beta_0\beta_1\beta_2^2 + 4m^2 b_2\beta_1^3\beta_2 + 12m^2 b_2\beta_0\beta_1\beta_2^2\\
- 3\Delta_0 \ln^2 k\beta_0 m\beta_1\chi + 24mb_2\beta_0\beta_1\beta_2^2 = 0,\\
2mb_2\beta_1^4 + 12mb_2\beta_0^2\beta_2^2 + 6b_2\beta_0^2\beta_2^2 + b_2\beta_1^4 - 4\ln^2 k\alpha\beta_2^2 + m^2 b_2\beta_1^4 - \Delta_2\beta_2^2 + 6m^2 b_2\beta_0^2\beta_2^2\\
+ 12b_2\beta_0\beta_1^2\beta_2 - m^2\Delta_2\beta_2^2 + 4\Delta_0 \ln^2 k\beta_2^2 - 2m\Delta_2\beta_2^2 + 12m^2 b_2\beta_0\beta_1^2\beta_2 + 24mb_2\beta_0\beta_1^2\beta_2 = 0,\\
-2m^2\Delta_2\beta_1\beta_2 - 2\Delta_2\beta_1\beta_2 - 4\ln^2 k\alpha\beta_1\beta_2 + 24mb_2\beta_0^2\beta_1\beta_2 + 12b_2\beta_0^2\beta_1\beta_2 + 5\Delta_0 \ln^2 k\beta_1\beta_2\\
+ 4m^2 b_2\beta_0\beta_1^3 + 4b_2\beta_0\beta_1^3 + 12m^2 b_2\beta_0^2\beta_1\beta_2 + 8mb_2\beta_0\beta_1^3 - 4m\Delta_2\beta_1\beta_2 + \Delta_0 \ln^2 k\beta_1 m\beta_2 = 0,\\
4b_2\beta_0^3\beta_2 + 4\Delta_0 \ln^2 k\beta_0 m\beta_2 - 2\Delta_2\beta_0\beta_2 + 6m^2 b_2\beta_0^2\beta_1^2 - \Delta_2\beta_1^2 - \ln^2 k\alpha\beta_1^2 + 12mb_2\beta_0^2\beta_1^2\\
- 4m\Delta_2\beta_0\beta_2 + 4m^2 b_2\beta_0^3\beta_2 - 2m^2\Delta_2\beta_0\beta_2 - 2m\Delta_2\beta_1^2 - m^2\Delta_2\beta_1^2 + \Delta_0 \ln^2 k\beta_1^2\\
+4\Delta_0 \ln^2 k\beta_0\beta_2 + 8mb_2\beta_0^3\beta_2 + 6b_2\beta_0^2\beta_1^2 = 0,\\
4b_2\beta_0^3\beta_1 + \Delta_0 \ln^2 k\beta_0\beta_1 - 2\Delta_2\beta_0\beta_1 - 2m^2\Delta_2\beta_0\beta_1 - 4m\Delta_2\beta_0\beta_1 + 8mb_2\beta_0^3\beta_1\\
+4m^2 b_2\beta_0^3\beta_1 + \Delta_0 \ln^2 k\beta_0 m\beta_1 = 0,\\
-2m\Delta_2\beta_0^2 + 2mb_2\beta_0^4 - m^2\Delta_2\beta_0^2 + m^2 b_2\beta_0^4 + m^2 b_3 + 2mb_3 + b_3 - \Delta_2\beta_0^2 + b_2\beta_0^4 = 0.
\end{gathered} \tag{324}$$

On solving the above algebraic Eqs. (324) by using the Maple, one gets the following results:

$$\begin{gathered}
\beta_0 = \frac{\epsilon}{\ln k}\sqrt{\frac{(1+m)b_3}{3\Delta_0}},\quad \beta_1 = 0,\quad \beta_2 = \frac{\epsilon}{\ln k}\sqrt{\frac{2(1+m)\chi b_3}{3\Delta_0}},\quad \alpha = \frac{1}{2}(m+3)\Delta_0,\\
\Delta_2 = \frac{4\Delta_0 \ln^2 k}{1+m},\quad b_2 = \frac{3\Delta_0^2 \ln^4 k}{(1+m)^2 b_3},
\end{gathered} \tag{325}$$

provided $\Delta_0 b_3 > 0$, $\chi > 0$ and $\epsilon = \pm 1$.

Substituting (325) along with (33) into Eq. (119), one gets the solutions of Eq. (297) in the form:

$$q(x,t) = \left\{\frac{1}{\ln k}\sqrt{\frac{(1+m)b_3}{3\Delta_0}}\left[1+\frac{4\epsilon A\sqrt{2\chi}}{4A^2\exp_k\left[2(x-ct)\right]+\chi\exp_k\left[-2(x-ct)\right]}\right]\right\}^{\frac{1}{m+1}} e^{i(-\kappa x+\omega t+\theta_0)}. \tag{326}$$

In particular, if we set $\chi = 4A^2$ in (326), then we have the bright soliton solution of Eq. (297) as:

$$q(x,t) = \left\{\frac{1}{\ln k}\sqrt{\frac{(1+m)b_3}{3\Delta_0}}\left[1+\epsilon\sqrt{2}\operatorname{sech}\left[2(x-ct)\ln k\right]\right]\right\}^{\frac{1}{m+1}} e^{i(-\kappa x+\omega t+\theta_0)}. \tag{327}$$

Similarly, we can find many other solutions by choosing other values for p and N.

13. QUADRATIC-CUBIC LAW

For the quadratic-cubic law nonlinearity, we have

$$F(\phi) = b_1\sqrt{\phi} + b_2\phi, \tag{328}$$

where b_1 and b_2 are arbitrary constants.

Equation (1) corresponding to quadratic-cubic law nonlinearity (328) is given by:

$$\begin{aligned} &iq_t + iaq_{xxx} + \left(b_1|q| + b_2|q|^2\right)q = \alpha\frac{|q_x|^2}{q^*} \\ &+ \frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \\ &+ i\left[\delta q_x + \lambda\left(|q|^{2m}q\right)_x + \mu\left(|q|^{2m}\right)_x q + \nu|q|^{2m}q_x\right], \end{aligned} \tag{329}$$

where Eq. (41) reduces to:

$$\begin{aligned} &\Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2 + b_1\phi^3 \\ &- \kappa(\lambda+\nu)\phi^{2m+2} + b_2\phi^4 = 0. \end{aligned} \tag{330}$$

For integrability, one must select $m = 1$. This leads to the modification of Eq. (1) corresponding to quadratic-cubic law nonlinearity as:

$$\begin{aligned} &iq_t + iaq_{xxx} + \left(b_1|q| + b_2|q|^2\right)q = \alpha\frac{|q_x|^2}{q^*} \\ &+ \frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \\ &+ i\left[\delta q_x + \lambda\left(|q|^2q\right)_x + \mu\left(|q|^2\right)_x q + \nu|q|^2q_x\right], \end{aligned} \tag{331}$$

Consequently, Eq. (330) changes to:

$$\begin{aligned} &\Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2 + b_1\phi^3 \\ &+ [b_2 - \kappa(\lambda+\nu)]\phi^4 = 0. \end{aligned} \tag{332}$$

In the next two subsections, we will solve Eq. (332) using the following two methods.

13.1. New Mapping Methos

According to the new mapping method, we balance $\phi\phi''$ with ϕ^4 in Eq. (332) yields the balance number $N = 1$. Now, from (6), the solution of Eq. (332) has the same form (48). Substituting (48) along with (7) into Eq. (332), collecting all the coefficients of $F^l(\xi)[F'(\xi)]^j$ $(l = 0,1,\ldots,8,\ j = 0,1)$ and setting them to zero, we have the following algebraic equations:

$$\begin{aligned} &-\frac{4}{3}\alpha\delta_2^2 s + [b_2 - \kappa(\lambda+\nu)]\delta_2^4 + \frac{8}{3}\Delta_0\delta_2^2 s = 0, \\ &4[b_2 - \kappa(\lambda+\nu)]\delta_1\delta_2^3 \\ &+ \frac{11}{3}\Delta_0\delta_1\delta_2 s - \frac{4}{3}\alpha\delta_1\delta_2 s = 0, \\ &-\frac{1}{3}\alpha\delta_1^2 s + b_1\delta_2^3 + 6[b_2 - \kappa(\lambda+\nu)]\delta_1^2\delta_2^2 \\ &+ 4[b_2 - \kappa(\lambda+\nu)]\delta_0\delta_2^3 - 2\alpha\delta_2^2 h + \Delta_0\delta_1^2 s \\ &+ 3\Delta_0\delta_2^2 h + \frac{8}{3}\Delta_0\delta_0\delta_2 s = 0, \\ &4\Delta_0\delta_1\delta_2 h - 2\alpha\delta_1\delta_2 h + 3b_1\delta_1\delta_2^2 \\ &+ 4\{\delta_1^3\delta_2 + 3\delta_0\delta_1\delta_2^2\}[b_2 - \kappa(\lambda+\nu)] + \Delta_0\delta_0\delta_1 s = 0, \\ &-4\alpha\delta_2^2 p - \Delta_2\delta_2^2 + 6[b_2 - \kappa(\lambda+\nu)]\delta_0^2\delta_2^2 \\ &- \frac{1}{2}\alpha\delta_1^2 h + 4\Delta_0\delta_2^2 p + 3b_1\delta_0\delta_2^2 + \Delta_0\delta_1^2 h \\ &+12[b_2 - \kappa(\lambda+\nu)]\delta_0\delta_1^2\delta_2 + 3b_1\delta_1^2\delta_2 \\ &+ [b_2 - \kappa(\lambda+\nu)]\delta_1^4 + 3\Delta_0\delta_0\delta_2 h = 0, \\ &\Delta_0\delta_0\delta_1 h + 12[b_2 - \kappa(\lambda+\nu)]\delta_0^2\delta_1\delta_2 \\ &+ 6b_1\delta_0\delta_1\delta_2 + 5\Delta_0\delta_1\delta_2 p - 2\Delta_2\delta_1\delta_2 \\ &- 4\alpha\delta_1\delta_2 p + b_1\delta_1^3 + 4[b_2 - \kappa(\lambda+\nu)]\delta_0\delta_1^3 = 0, \\ &6[b_2 - \kappa(\lambda+\nu)]\delta_0^2\delta_1^2 - \alpha\delta_1^2 p + \Delta_0\delta_1^2 p \\ &+ 4\Delta_0\delta_0\delta_2 p - 2\Delta_2\delta_0\delta_2 + 4[b_2 - \kappa(\lambda+\nu)]\delta_0^3\delta_2 \\ &+3b_1\delta_0\delta_1^2 - 4\alpha\delta_2^2 r + 2\Delta_0\delta_2^2 r + 3b_1\delta_0^2\delta_2 - \Delta_2\delta_1^2 = 0, \\ &\Delta_0\delta_0\delta_1 p - 4\alpha\delta_1\delta_2 r - 2\Delta_2\delta_0\delta_1 + 3b_1\delta_0^2\delta_1 \\ &+ 2\Delta_0\delta_1\delta_2 r + 4[b_2 - \kappa(\lambda+\nu)]\delta_0^3\delta_1 = 0, \\ &-\alpha r\delta_1^2 + [b_2 - \kappa(\lambda+\nu)]\delta_0^4 \\ &- \Delta_2\delta_0^2 + b_1\delta_0^3 + 2\Delta_0\delta_0\delta_2 r = 0. \end{aligned} \tag{333}$$

With the aid of the solutions (8)–(29), we have the following types of solutions:

Type 1. Substituting $s = \dfrac{3h^2}{16p}$, $r = \dfrac{16p^2}{27h}$, into the algebraic equations (333) and solve them by Maple, one gets the following results:

$$\begin{aligned} &\delta_0 = \frac{\epsilon}{3}\sqrt{\frac{\Delta_2}{b_2 - \kappa(\lambda+\nu)}}, \quad \delta_1 = 0, \\ &\delta_2 = \frac{\epsilon h\Delta_0}{2\Delta_2}\sqrt{\frac{\Delta_2}{b_2 - \kappa(\lambda+\nu)}}, \quad p = \frac{\Delta_2}{2\Delta_0}, \\ &h = h, \quad \alpha = \frac{5}{2}\Delta_0, \\ &b_1 = \frac{4\epsilon[b_2 - \kappa(\lambda+\nu)]}{3}\sqrt{\frac{\Delta_2}{b_2 - \kappa(\lambda+\nu)}}, \end{aligned} \tag{334}$$

provided $\Delta_2[b_2 - \kappa(\lambda+\nu)] > 0$, $\Delta_0 \neq 0$ and $\epsilon = \pm 1$.

If we substitute (334) along with (8)–(12) into Eq. (48), then Eq. (331) has the following solutions.

13.1.1. Soliton solutions.

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b_2-\kappa(\lambda+\nu)}}\times\left[\frac{1-\tanh^2\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}{3+\tanh^2\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)},\tag{335}$$

and

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b_2-\kappa(\lambda+\nu)}}\times\left[\frac{1-\coth^2\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}{3+\coth^2\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)},\tag{336}$$

provided $\Delta_2[b_2-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm 1$.

13.1.2. Periodic solutions.

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b_2-\kappa(\lambda+\nu)}}\times\left[\frac{1+\tan^2\left(\epsilon\sqrt{\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}{3-\tan^2\left(\epsilon\sqrt{\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)},\tag{337}$$

and

$$q(x,t)=\epsilon\sqrt{\frac{\Delta_2}{b_2-\kappa(\lambda+\nu)}}\times\left[\frac{1-\cot^2\left(\epsilon\sqrt{\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}{3+\cot^2\left(\epsilon\sqrt{\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)},\tag{338}$$

provided $\Delta_2[b_2-\kappa(\lambda+\nu)]>0$, $\Delta_0\Delta_2>0$ and $\epsilon=\pm 1$.

Type 2. Substituting $s=\frac{3h^2}{16p}$, $r=0$, into the algebraic Eqs. (333) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=\frac{\epsilon}{3}\sqrt{-\frac{3\Delta_2}{b_2-\kappa(\lambda+\nu)}},\quad \delta_1=0,\\ \delta_2=-\frac{\epsilon h\Delta_0}{2\Delta_2}\sqrt{-\frac{3\Delta_2}{b_2-\kappa(\lambda+\nu)}},\\ p=-\frac{\Delta_2}{6\Delta_0},\quad h=h,\quad \alpha=\frac{5}{2}\Delta_0,\\ b_1=-\frac{4\epsilon[b_2-\kappa(\lambda+\nu)]}{3}\sqrt{-\frac{3\Delta_2}{b_2-\kappa(\lambda+\nu)}},\end{gathered}\tag{339}$$

provided $\Delta_2[b_2-\kappa(\lambda+\nu)]<0$, $\Delta_0\neq 0$ and $\epsilon=\pm 1$.

If we substitute (339) along with (14) and (15) into Eq. (48), then Eq. (331) has the following solutions:

13.1.3. Dark and singular solitons.

$$q(x,t)=\frac{\epsilon}{6}\sqrt{-\frac{3\Delta_2}{b_2-\kappa(\lambda+\nu)}}\times\left[1+\tanh\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)\right]e^{i(-\kappa x+\omega t+\theta_0)},\tag{340}$$

and

$$q(x,t)=\frac{\epsilon}{6}\sqrt{-\frac{3\Delta_2}{b_2-\kappa(\lambda+\nu)}}\times\left[1+\coth\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)\right]e^{i(-\kappa x+\omega t+\theta_0)},\tag{341}$$

respectively, provided $\Delta_2[b_2-\kappa(\lambda+\nu)]<0, \Delta_0\Delta_2<0$ and $\epsilon=\pm 1$.

Type 3. Substituting $r=0$, into the algebraic equations (333) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=\frac{\epsilon}{3}\sqrt{-\frac{3\Delta_2}{b_2-\kappa(\lambda+\nu)}},\quad \delta_1=0,\\ \delta_2=-\frac{\epsilon h\Delta_0}{2\Delta_2}\sqrt{-\frac{3\Delta_2}{b_2-\kappa(\lambda+\nu)}},\quad p=-\frac{\Delta_2}{6\Delta_0},\\ s=-\frac{9h^2\Delta_0}{8\Delta_2},\quad h=h,\quad \alpha=\frac{5}{2}\Delta_0,\\ b_1=-\frac{4\epsilon[b_2-\kappa(\lambda+\nu)]}{3}\sqrt{-\frac{3\Delta_2}{b_2-\kappa(\lambda+\nu)}},\end{gathered}\tag{342}$$

provided $\Delta_2[b_2-\kappa(\lambda+\nu)]<0, \Delta_0\neq 0$ and $\epsilon=\pm 1$.

If we substitute (342) along with (16)–(29) into Eq. (48), then Eq. (331) has the following solutions.

13.1.4. Soliton solutions.

$$q(x,t)=\epsilon\sqrt{-\frac{\Delta_2}{3[b_2-\kappa(\lambda+\nu)]}}\times\left[1-\frac{2\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}{4-\left[1+\tanh\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)\right]^2}\right]e^{i(-\kappa x+\omega t+\theta_0)},\tag{343}$$

$$q(x,t)=\epsilon\sqrt{-\frac{\Delta_2}{3[b_2-\kappa(\lambda+\nu)]}}\times\left[1+\frac{2\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}{4-\left[1+\coth\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)\right]^2}\right]e^{i(-\kappa x+\omega t+\theta_0)},\tag{344}$$

$$q(x,t)=\epsilon\sqrt{-\frac{\Delta_2}{3[b_2-\kappa(\lambda+\nu)]}}\times\left[1-\frac{\operatorname{sech}^2\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}{2+2\tanh\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (345)$$

$$q(x,t)=\epsilon\sqrt{-\frac{\Delta_2}{3[b_2-\kappa(\lambda+\nu)]}}\times\left[1+\frac{\operatorname{cosech}^2\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}{2+2\coth\left(\epsilon\sqrt{-\frac{\Delta_2}{6\Delta_0}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \quad (346)$$

provided $\Delta_2[b_2-\kappa(\lambda+\nu)]<0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

13.2. Addendum to Kudryashov's Method

According to this method, we balance $\phi\phi''$ with ϕ^4 in Eq. (332), one gets the same relation (70). Now, we will discuss the following cases:

Case 1. If we choose $p=1$, then $N=1$. Thus, we deduce that from (31) that Eq. (332) has the same formal solution (71). Substituting (71) along with (32) into Eq. (332), collecting all the coefficients of each power of $[R(\xi)]^{m_1}\,[R'(\xi)]^j$, $(m_1=0,1,2,\ldots,4,\ j=0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}\alpha\beta_1^2\chi\ln^2k-2\Delta_0\beta_1^2\chi\ln^2k+[b_2-\kappa(\lambda+\nu)]\beta_1^4=0,\\ b_1\beta_1^3-2\Delta_0\beta_1\beta_0\chi\ln^2k+4[b_2-\kappa(\lambda+\nu)]\beta_0\beta_1^3=0,\\ -\alpha\beta_1^2\ln^2k+3b_1\beta_0\beta_1^2-\Delta_2\beta_1^2\\ +\Delta_0\beta_1^2\ln^2k+6[b_2-\kappa(\lambda+\nu)]\beta_0^2\beta_1^2=0,\\ -2\Delta_2\beta_0\beta_1+4[b_2-\kappa(\lambda+\nu)]\beta_0^3\beta_1\\ +3b_1\beta_0^2\beta_1+\Delta_0\beta_1\beta_0\ln^2k=0,\\ +[b_2-\kappa(\lambda+\nu)]\beta_0^4+b_1\beta_0^3-\Delta_2\beta_0^2=0.\end{gathered} \quad (347)$$

On solving the above algebraic Eqs. (347) by using the Maple, one gets the following results:

$$\begin{gathered}\beta_0=\frac{\epsilon\ln k}{2}\sqrt{\frac{6\Delta_0}{b_2-\kappa(\lambda+\nu)}},\\ \beta_1=\frac{\epsilon\ln k}{2}\sqrt{\frac{6\chi\Delta_0}{b_2-\kappa(\lambda+\nu)}},\\ b_1=-\frac{4\epsilon[b_2-\kappa(\lambda+\nu)]\ln k}{3}\sqrt{\frac{6\Delta_0}{b_2-\kappa(\lambda+\nu)}},\\ \Delta_2=-\frac{5}{2}\Delta_0\ln^2k,\quad \alpha=\frac{1}{2}\Delta_0,\end{gathered} \quad (348)$$

provided $(b_2-\kappa(\lambda+\nu))\Delta_0>0,\chi>0$ and $\epsilon=\pm1$.

Substituting (348) along with (33) into Eq. (71), one gets the solutions of Eq. (331) in the form:

$$q(x,t)=\frac{\epsilon\ln k}{2}\sqrt{\frac{6\Delta_0}{b_2-\kappa(\lambda+\nu)}}\left[1+\frac{4\epsilon A\sqrt{\chi}}{4A^2\exp_k[(x-ct)]+\chi\exp_k[-(x-ct)]}\right]e^{i(-\kappa x+\omega t+\theta_0)}. \quad (349)$$

In particular, if we set $\chi=4A^2$ in (349), then we have the bright soliton solution of Eq. (331) as:

$$q(x,t)=\frac{\epsilon\ln k}{2}\sqrt{\frac{6\Delta_0}{b_2-\kappa(\lambda+\nu)}}\times\{1+\epsilon\operatorname{sech}[(x-ct)\ln k]\}e^{i(-\kappa x+\omega t+\theta_0)}. \quad (350)$$

Case 2. If we choose $p=2$, then $N=2$. Thus, we deduce that from (31) that Eq. (332) has the same formal solution (77). Substituting (77) along with Eq. (32) into Eq. (332), collecting all the coefficients of each power of $[R(\xi)]^{m_2}\,[R'(\xi)]^j$, $(m_2=0,1,2,\ldots,8,\ j=0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}[b_2-\kappa(\lambda+\nu)]\beta_2^4+4\alpha\chi\beta_2^2\ln^2k\\ -8\Delta_0\beta_2^2\chi\ln^2k=0,\\ 4[b_2-\kappa(\lambda+\nu)]\beta_1\beta_2^3+4\alpha\chi\beta_1\beta_2\ln^2k\\ -11\Delta_0\beta_1\beta_2\chi\ln^2k=0,\\ -3\Delta_0\beta_1^2\chi\ln^2k+\alpha\beta_1^2\chi\ln^2k-8\Delta_0\beta_0\beta_2\chi\ln^2k\\ +b_1\beta_2^3+2[b_2-\kappa(\lambda+\nu)]\left[3\beta_1^2\beta_2^2+2\beta_0\beta_2^3\right]=0,\\ -3\Delta_0\beta_1\beta_0\chi\ln^2k+12[b_2-\kappa(\lambda+\nu)]\beta_0\beta_1\beta_2^2\\ +3b_1\beta_1\beta_2^2+4[b_2-\kappa(\lambda+\nu)]\beta_1^3\beta_2=0,\\ -\Delta_2\beta_2^2+4\Delta_0\beta_2^2\ln^2k+3b_1\beta_0\beta_2^2\\ +3b_1\beta_1^2\beta_2+6[b_2-\kappa(\lambda+\nu)]\beta_0^2\beta_2^2\\ +12[b_2-\kappa(\lambda+\nu)]\beta_0\beta_1^2\beta_2-4\alpha\beta_2^2\ln^2k\\ +[b_2-\kappa(\lambda+\nu)]\beta_1^4=0,\\ 6b_1\beta_0\beta_1\beta_2-2\Delta_2\beta_1\beta_2+b_1\beta_1^3\\ +12[b_2-\kappa(\lambda+\nu)]\beta_0^2\beta_1\beta_2+4[b_2-\kappa(\lambda+\nu)]\beta_0\beta_1^3\\ +5\Delta_0\beta_1\beta_2\ln^2k-4\alpha\beta_1\beta_2\ln^2k=0,\\ -\Delta_2\beta_1^2-\alpha\beta_1^2\ln^2k+4\Delta_0\beta_0\beta_2\ln^2k+3b_1\beta_0^2\beta_2\end{gathered} \quad (351)$$

$$+\Delta_0\beta_1^2\ln^2 k+4[b_2-\kappa(\lambda+\nu)]\beta_0^3\beta_2$$
$$+6[b_2-\kappa(\lambda+\nu)]\beta_0^2\beta_1^2+3b_1\beta_0\beta_1^2-2\Delta_2\beta_0\beta_2=0,$$
$$-2\Delta_2\beta_0\beta_1+4[b_2-\kappa(\lambda+\nu)]\beta_0^3\beta_1$$
$$+3b_1\beta_0^2\beta_1+\Delta_0\beta_1\beta_0\ln^2 k=0,$$
$$-\Delta_2\beta_0^2+b_1\beta_0^3+[b_2-\kappa(\lambda+\nu)]=0.$$

On solving the above algebraic Eqs. (351) by using the Maple, one gets the following results:

$$\beta_0=\epsilon\sqrt{\frac{6\Delta_0}{b_2-\kappa(\lambda+\nu)}}\ln k,$$
$$\beta_1=\epsilon\sqrt{\frac{6\chi\Delta_0}{b_2-\kappa(\lambda+\nu)}}\ln k,$$
$$b_1=-\frac{8\epsilon[b_2-\kappa(\lambda+\nu)]\ln k}{3}\sqrt{\frac{6\Delta_0}{b_2-\kappa(\lambda+\nu)}}, \quad (352)$$
$$\Delta_2=-10\Delta_0\ln^2 k,\quad \alpha=\frac{1}{2}\Delta_0,$$

provided $(b_2-\kappa(\lambda+\nu))\Delta_0>0, \chi>0$ and $\epsilon=\pm 1$.

Substituting (352) along with (33) into Eq. (71), one gets the solutions of Eq. (331) in the form:

$$q(x,t)=\epsilon\ln k\sqrt{\frac{6\Delta_0}{b_2-\kappa(\lambda+\nu)}}\left[1+\frac{4\epsilon A\sqrt{\chi}}{4A^2\exp_k[2(x-ct)]+\chi\exp_k[-2(x-ct)]}\right]e^{i(-\kappa x+\omega t+\theta_0)}. \quad (353)$$

In particular, if we set $\chi=4A^2$ in (353), then we have the bright soliton solution of Eq. (331) as:

$$q(x,t)=\epsilon\ln k\sqrt{\frac{6\Delta_0}{b_2-\kappa(\lambda+\nu)}} \quad (354)$$
$$\times\{1+\epsilon\operatorname{sech}[2(x-ct)\ln k]\}e^{i(-\kappa x+\omega t+\theta_0)}.$$

Similarly, we can find many other solutions by choosing other values for p and N.

14. PARABOLIC NON-LOCAL LAW

For such nonlinearity, we have

$$F(\phi)=b_1\phi+b_2\phi^2+b_3(\phi)_{xx}, \quad (355)$$

where b_1, b_2 and b_3 are constants, such that the coefficients of b_1 and b_2 constitute parabolic law while the coefficient of b_3 stems from non-local nonlinearity.

Equation (1) corresponding to parabolic–non-local law nonlinearity (355) is given by:

$$iq_t+iaq_{xxx}+\left[b_1|q|^2+b_2|q|^4+b_3\left(|q|^2\right)_{xx}\right]q=\alpha\frac{|q_x|^2}{q^*}$$
$$+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q \quad (356)$$
$$+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_x q+\nu|q|^{2m}q_x\right],$$

where Eq. (41) reduces to:

$$\Delta_0\phi\phi''-\alpha\phi'^2+2b_3\left(\phi^3\phi''+\phi^2\phi'^2\right)$$
$$-\Delta_2\phi^2+b_1\phi^4-\kappa(\lambda+\nu)\phi^{2m+2}+b_2\phi^6=0. \quad (357)$$

For integrability, one must select $m=1$. This leads to the modification of Eq. (1) corresponding to parabolic–non-local law nonlinearity as:

$$iq_t+iaq_{xxx}+\left[b_1|q|^2+b_2|q|^4+b_3\left(|q|^2\right)_{xx}\right]q=\alpha\frac{|q_x|^2}{q^*}$$
$$+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q \quad (358)$$
$$+i\left[\delta q_x+\lambda\left(|q|^2q\right)_x+\mu\left(|q|^2\right)_x q+\nu|q|^2q_x\right].$$

Consequently, Eq. (357) becomes:

$$\Delta_0\phi\phi''-\alpha\phi'^2+2b_3\left(\phi^3\phi''+\phi^2\phi'^2\right)$$
$$-\Delta_2\phi^2+[b_1-\kappa(\lambda+\nu)]\phi^4+b_2\phi^6=0. \quad (359)$$

In the next two subsections, we will solve Eq. (332) using the following two methods.

14.1. New Mapping Method

According to the new mapping method, we balance $\phi^3\phi''$ with φ^6 in Eq. (359) yields the balance number $N=1$. Now, from (6), the solution of Eq. (359) has the same form (48). Substituting (48) along with (7) into Eq. (359), collecting all the coefficients of $F^l(\xi)\left[F'(\xi)\right]^j$ $(l=0,1,\ldots,12, j=0,1)$ and setting them to zero, we have the following algebraic equations:

$$b_2\delta_2^6+8b_3\delta_2^4s=0,$$
$$26b_3\delta_1\delta_2^3s+6b_2\delta_1\delta_2^5=0,$$
$$\frac{92}{3}b_3\delta_1^2s\delta_2^2+10b_3\delta_2^4h+6b_2\delta_0\delta_2^5$$
$$+\frac{64}{3}b_3\delta_2^3s\delta_0+15b_2\delta_1^2\delta_2^4=0,$$
$$32b_3\delta_1\delta_2^3h+\frac{46}{3}b_3\delta_1^3s\delta_2+30b_2\delta_0\delta_1\delta_2^4$$
$$+\frac{146}{3}b_3\delta_1\delta_2^2s\delta_0+20b_2\delta_1^3\delta_2^3=0,$$

$$37b_3\delta_1^2h\delta_2^2+\frac{56}{3}b_3\delta_2^2s\delta_0^2+\frac{8}{3}\Delta_0\delta_2^2s$$
$$+[b_1-\kappa(\lambda+\nu)]\delta_2^4+\frac{8}{3}b_3\delta_1^4s$$
$$+16b_3\delta_2^4p+15b_2\delta_0^2\delta_2^4-\frac{4}{3}\alpha\delta_2^2s+\frac{104}{3}b_3\delta_1^2s\delta_0\delta_2$$
$$+26b_3\delta_2^3h\delta_0+15b_2\delta_1^4\delta_2^2+60b_2\delta_0\delta_1^2\delta_2^3=0,$$
$$60b_2\delta_0\delta_1^3\delta_2^2+\frac{11}{3}\Delta_0\delta_1\delta_2s+6b_2\delta_1^5\delta_2$$
$$+\frac{74}{3}b_3\delta_1\delta_2s\delta_0^2+60b_2\delta_0^2\delta_1\delta_2^3+18b_3\delta_1^3h\delta_2$$
$$+4[b_1-\kappa(\lambda+\nu)]\delta_1\delta_2^3+\frac{22}{3}b_3\delta_1^3s\delta_0$$
$$+50b_3\delta_1\delta_2^3p+58b_3\delta_1\delta_2^2h\delta_0-\frac{4}{3}\alpha\delta_1\delta_2s=0,$$
$$56b_3\delta_1^2p\delta_2^2+20b_2\delta_0^3\delta_2^3+\frac{8}{3}\Delta_0\delta_0\delta_2s$$
$$+4[b_1-\kappa(\lambda+\nu)]\delta_0\delta_2^3-\frac{1}{3}\alpha\delta_1^2s+\Delta_0\delta_1^2s+40b_3\delta_2^3p\delta_0$$
$$+\frac{16}{3}b_3\delta_2s\delta_0^3+3b_3\delta_1^4h-2\alpha\delta_2^2h$$
$$+6[b_1-\kappa(\lambda+\nu)]\delta_1^2\delta_2^2+3\Delta_0\delta_2^2h$$
$$+b_2\delta_1^6+30b_2\delta_0\delta_1^4\delta_2+12b_3\delta_2^4r+22b_3\delta_2^2h\delta_0^2$$
$$+40b_3\delta_1^2h\delta_0\delta_2+90b_2\delta_0^2\delta_1^2\delta_2^2+(20/3)b_3\delta_1^2s\delta_0^2=0,$$
$$28b_3\delta_1\delta_2h\delta_0^2+26b_3\delta_1^3p\delta_2+6b_2\delta_0\delta_1^5$$
$$+36b_3\delta_1\delta_2^3r+4[b_1-\kappa(\lambda+\nu)]\delta_1^3\delta_2$$
$$+8b_3\delta_1^3h\delta_0-2\alpha\delta_1\delta_2h+2b_3\delta_1s\delta_0^3+\Delta_0\delta_0\delta_1s$$
$$+12[b_1-\kappa(\lambda+\nu)]\delta_0\delta_1\delta_2^2+4\Delta_0\delta_1\delta_2h \qquad (360)$$
$$+86b_3\delta_1\delta_2^2p\delta_0+60b_2\delta_0^2\delta_1^3\delta_2+60b_2\delta_0^3\delta_1\delta_2^2=0,$$
$$6b_3\delta_2h\delta_0^3-\frac{1}{2}\alpha\delta_1^2h+56b_3\delta_1^2p\delta_0\delta_2+15b_2\delta_0^2\delta_1^4$$
$$+4\Delta_0\delta_2^2p+7b_3\delta_1^2h\delta_0^2+38b_3r\delta_1^2\delta_2^2+\Delta_0\delta_1^2h$$
$$-4\alpha\delta_2^2p+3\Delta_0\delta_0\delta_2h+6[b_1-\kappa(\lambda+\nu)]\delta_0^2\delta_2^2$$
$$+4b_3\delta_1^4p+15b_2\delta_0^4\delta_2^2+32b_3\delta_2^2p\delta_0^2+28b_3\delta_2^3r\delta_0$$
$$+[b_1-\kappa(\lambda+\nu)]\delta_1^4+12[b_1-\kappa(\lambda+\nu)]\delta_0\delta_1^2\delta_2$$
$$-\Delta_2\delta_2^2+60b_2\delta_0^3\delta_1^2\delta_2=0,$$
$$30b_2\delta_0^4\delta_1\delta_2+4[b_1-\kappa(\lambda+\nu)]\delta_0\delta_1^3$$
$$+56b_3\delta_1\delta_2^2r\delta_0+10b_3\delta_1^3p\delta_0+20b_2\delta_0^3\delta_1^3$$
$$+5\Delta_0\delta_1\delta_2p+38b_3\delta_1\delta_2p\delta_0^2+\Delta_0\delta_0\delta_1h$$
$$-2\Delta_2\delta_1\delta_2+2b_3\delta_1h\delta_0^3+16b_3r\delta_1^3\delta_2$$
$$+12[b_1-\kappa(\lambda+\nu)]\delta_0^2\delta_1\delta_2-4\alpha\delta_1\delta_2p=0,$$
$$-4\alpha\delta_2^2r+32b_3r\delta_1^2\delta_0\delta_2+8b_3\delta_1^2p\delta_0^2+8b_3\delta_2p\delta_0^3$$
$$-2\Delta_2\delta_0\delta_2+6[b_1-\kappa(\lambda+\nu)]\delta_0^2\delta_1^2$$
$$-\alpha\delta_1^2p+4\Delta_0\delta_0\delta_2p+2\Delta_0\delta_2^2r+2b_3r\delta_1^4$$
$$+4[b_1-\kappa(\lambda+\nu)]\delta_0^3\delta_2+\Delta_0\delta_1^2p+15b_2\delta_0^4\delta_1^2$$
$$+6b_2\delta_0^5\delta_2+20b_3\delta_2^2r\delta_0^2-\Delta_2\delta_1^2=0,$$
$$\Delta_0\delta_0\delta_1p+4[b_1-\kappa(\lambda+\nu)]\delta_0^3\delta_1+20b_3\delta_1\delta_2r\delta_0^2$$
$$+4b_3r\delta_1^3\delta_0-2\Delta_2\delta_0\delta_1+2\Delta_0\delta_1\delta_2r$$
$$+2b_3\delta_1p\delta_0^3-4\alpha\delta_1\delta_2r+6b_2\delta_0^5\delta_1=0,$$
$$-\Delta_2\delta_0^2-\alpha r\delta_1^2+b_2\delta_0^6+[b_1-\kappa(\lambda+\nu)]\delta_0^4$$
$$+2\Delta_0\delta_0\delta_2r+2b_3r\delta_1^2\delta_0^2+4b_3\delta_2r\delta_0^3=0.$$

With the aid of the solutions (8)–(29), we have the following types of solutions:

Type 1. Substituting $s=\frac{3h^2}{16p}, r=0$, into the algebraic Eqs. (360) and solve them by Maple, we get the following results:

$$\delta_0=\frac{\epsilon}{2}\sqrt{-\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{b_2b_3}},\quad \delta_1=0,\quad \delta_2=\frac{6\epsilon hb_3^2}{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}\sqrt{-\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{b_2b_3}}, \qquad (361)$$
$$p=\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2},\quad h=h,\quad \alpha=0,\quad \Delta_2=-\frac{\{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0\}\{b_3[b_1-\kappa(\lambda+\nu)]+b_2\Delta_0\}}{16b_2b_3^2},$$

provided $\{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0\}b_2b_3<0$ and $\epsilon=\pm1$.

If we substitute (361) along with (14) and (15) into Eq. (48), then Eq. (358) has the following solutions.

14.1.1. Dark and singular solitons.

$$q(x,t)=\frac{\epsilon}{2}\sqrt{-\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{b_2b_3}}\tanh\left(\epsilon\sqrt{\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2}}(x-ct)\right)e^{i(-\kappa x+\omega t+\theta_0)}, \qquad (362)$$

and

$$q(x,t)=\frac{\epsilon}{2}\sqrt{-\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{b_2b_3}}\coth\left(\epsilon\sqrt{\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2}}(x-ct)\right)e^{i(-\kappa x+\omega t+\theta_0)}, \qquad (363)$$

respectively, provided $3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0>0$, $b_2b_3<0$ and $\epsilon=\pm1$.

Type 2. Substituting $r = 0$, into the algebraic Eqs. (360) and solve them by Maple, we get the following results:

$$\delta_0 = \frac{\epsilon}{2}\sqrt{-\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{b_2b_3}},\quad \delta_1 = 0,\quad \delta_2 = \frac{6\epsilon h b_3^2}{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}\sqrt{-\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{b_2b_3}},$$

$$p = \frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2},\quad s = \frac{9h^2b_3^2}{2\{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0\}},\quad h = h, \tag{364}$$

$$\alpha = 0,\quad \Delta_2 = -\frac{\{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0\}\{b_3[b_1-\kappa(\lambda+\nu)]+b_2\Delta_0\}}{16b_2b_3^2},$$

provided $\{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0\}b_2b_3 < 0$ and $\epsilon = \pm 1$.

If we substitute (364) along with (16)–(29) into Eq. (48), then Eq. (358) has the following solutions.

14.1.2. Soliton solutions.

$$q(x,t) = \frac{\epsilon}{2}\sqrt{-\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{b_2b_3}} \times\left[1-\frac{4\operatorname{sech}^2\left(\epsilon\sqrt{\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2}}(x-ct)\right)}{4-\left[1+\tanh\left(\epsilon\sqrt{\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2}}(x-ct)\right)\right]^2}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{365}$$

$$q(x,t) = \frac{\epsilon}{2}\sqrt{-\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{b_2b_3}} \times\left[1+\frac{4\operatorname{cosech}^2\left(\epsilon\sqrt{\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2}}(x-ct)\right)}{4-\left[1+\coth\left(\epsilon\sqrt{\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2}}(x-ct)\right)\right]^2}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{366}$$

$$q(x,t) = \frac{\epsilon}{2}\sqrt{-\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{b_2b_3}} \times\left[1-\frac{\operatorname{sech}^2\left(\epsilon\sqrt{\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2}}(x-ct)\right)}{1+\tanh\left(\epsilon\sqrt{\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{367}$$

$$q(x,t) = \frac{\epsilon}{2}\sqrt{-\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{b_2b_3}}\left[1+\frac{\operatorname{cosech}^2\left(\epsilon\sqrt{\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2}}(x-ct)\right)}{1+\coth\left(\epsilon\sqrt{\frac{3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0}{24b_3^2}}(x-ct)\right)}\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{368}$$

provided $3b_3[b_1-\kappa(\lambda+\nu)]-b_2\Delta_0 > 0, b_2b_3 < 0$ and $\epsilon = \pm 1$.

14.2. Addendum to Kudryashov's Method

According to this method, we balance $\phi^3\phi''$ with ϕ^6 in Eq. (359), one gets the same relation (70). Now, we will discuss the following cases:

Case 1. If we choose $p = 1$, then $N = 1$. Thus, we deduce that from (31) that Eq. (359) has the same formal solution (71). Substituting (71) along with (32) into Eq. (359), collecting all the coefficients of each power of $[R(\xi)]^{m_1}\,[R'(\xi)]^j$, $(m_1 = 0,1,2,\ldots,6,\ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
b_2\beta_1^6 - 6b_3\beta_1^4\chi\ln^2 k = 0, \quad 6b_2\beta_0\beta_1^5 - 16b_3\beta_1^3\chi\beta_0\ln^2 k = 0, \\
15b_2\beta_0^2\beta_1^4 + 4b_3\beta_1^4\ln^2 k - 14b_3\beta_1^2\chi\beta_0^2\ln^2 k + \alpha\beta_1^2\chi\ln^2 k - 2\Delta_0\beta_1^2\chi\ln^2 k + [b_1 - \kappa(\lambda+\nu)]\beta_1^4 = 0, \\
10b_3\beta_1^3\beta_0\ln^2 k + 4[b_1 - \kappa(\lambda+\nu)]\beta_0\beta_1^3 - 2\Delta_0\beta_1\beta_0\chi\ln^2 k + 20b_2\beta_0^3\beta_1^3 - 4b_3\beta_1\chi\beta_0^3\ln^2 k = 0, \\
-\Delta_2\beta_1^2 + \Delta_0\beta_1^2\ln^2 k + 6[b_1 - \kappa(\lambda+\nu)]\beta_0^2\beta_1^2 + 15b_2\beta_0^4\beta_1^2 + 8b_3\beta_1^2\beta_0^2\ln^2 k - \alpha\beta_1^2\ln^2 k = 0, \\
-2\Delta_2\beta_0\beta_1 + 4[b_1 - \kappa(\lambda+\nu)]\beta_0^3\beta_1 + \Delta_0\beta_1\beta_0\ln^2 k + 2b_3\beta_1\beta_0^3\ln^2 k + 6b_2\beta_0^5\beta_1 = 0, \\
-\Delta_2\beta_0^2 + [b_1 - \kappa(\lambda+\nu)]\beta_0^4 + b_2\beta_0^6 = 0.
\end{gathered} \tag{369}$$

On solving the above algebraic Eqs. (369) by using the Maple, one gets the following results:

$$\begin{gathered}
\beta_0 = 0, \quad \beta_1 = \epsilon\sqrt{\frac{6\chi b_3}{b_2}}\ln k, \\
\Delta_2 = \frac{\left[-b_2\Delta_0 + 6b_3[b_1 - \kappa(\lambda+\nu)] + 24b_3^2\ln^2 k\right]\ln^2 k}{b_2}, \\
\alpha = \frac{2\left[b_2\Delta_0 - 3b_3[b_1 - \kappa(\lambda+\nu)] - 12b_3^2\ln^2 k\right]}{b_2},
\end{gathered} \tag{370}$$

provided $\chi b_2 b_3 > 0$ and $\epsilon = \pm 1$.

Substituting (370) along with (33) into Eq. (71), one gets the solutions of Eq. (358) in the form:

$$q(x,t) = \epsilon\sqrt{\frac{6\chi b_3}{b_2}}(\ln k)\left[\frac{4A}{4A^2\exp_k[(x-ct)] + \chi\exp_k[-(x-ct)]}\right]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{371}$$

In particular, if we set $\chi = 4A^2$ in (371), then we have the bright soliton solution of Eq. (358) as:

$$q(x,t) = \epsilon\sqrt{\frac{6b_3}{b_2}}(\ln k)\operatorname{sech}[(x-ct)\ln k]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{372}$$

while, if $\chi = -4A^2$, one gets the singular soliton solution of Eq. (358) as:

$$q(x,t) = \epsilon\sqrt{-\frac{6b_3}{b_2}}(\ln k)\operatorname{cosech}[(x-ct)\ln k]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{373}$$

Case 2. If we choose $p = 2$, then $N = 2$. Thus, we deduce that from (31) that Eq. (359) has the same formal solution (77). Substituting (77) along with Eq. (32) into Eq. (359), collecting all the coefficients of each power of $[R(\xi)]^{m_2}$ $[R'(\xi)]^j$, $(m_2 = 0,1,2,\ldots,8,\ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
b_2\beta_2^6 - 24b_3\chi\beta_2^4\ln^2 k = 0, \\
-78b_3\chi\beta_1\beta_2^3\ln^2 k + 6b_2\beta_1\beta_2^5 = 0, \\
6b_2\beta_0\beta_2^5 + 15b_2\beta_1^2\beta_2^4 - 64b_3\chi\beta_2^3\beta_0\ln^2 k - 92b_3\chi\beta_1^2\beta_2^2\ln^2 k = 0, \\
-46b_3\chi\beta_1^3\beta_2\ln^2 k + 20b_2\beta_1^3\beta_2^3 - 146b_3\chi\beta_1\beta_2^2\beta_0\ln^2 k + 30b_2\beta_0\beta_1\beta_2^4 = 0, \\
15b_2\beta_1^4\beta_2^2 + [b_1 - \kappa(\lambda+\nu)]\beta_2^4 + 4\alpha\chi\beta_2^2\ln^2 k - 8\Delta_0\beta_2^2\chi\ln^2 k - 8b_3\beta_1^4\chi\ln^2 k + 15b_2\beta_0^2\beta_2^4 \\
-56b_3\chi\beta_2^2\beta_0^2\ln^2 k - 104b_3\chi\beta_1^2\beta_0\beta_2\ln^2 k + 60b_2\beta_0\beta_1^2\beta_2^3 + 16b_3\beta_2^4\ln^2 k = 0, \\
50b_3\beta_1\beta_2^3\ln^2 k - 11\Delta_0\beta_1\beta_2\chi\ln^2 k - 22b_3\chi\beta_1^3\beta_0\ln^2 k + 4[b_1 - \kappa(\lambda+\nu)]\beta_1\beta_2^3 + 6b_2\beta_1^5\beta_2 \\
-74b_3\chi\beta_1\beta_2\beta_0^2\ln^2 k + 4\alpha\chi\beta_1\beta_2\ln^2 k + 60b_2\beta_0^2\beta_1\beta_2^3 + 60b_2\beta_0\beta_1^3\beta_2^2 = 0, \\
-8\Delta_0\beta_0\beta_2\chi\ln^2 k + 40b_3\beta_2^3\beta_0\ln^2 k + 56b_3\beta_1^2\beta_2^2\ln^2 k + 6[b_1 - \kappa(\lambda+\nu)]\beta_1^2\beta_2^2 + 90b_2\beta_0^2\beta_1^2\beta_2^2 \\
-16b_3\beta_2\chi\beta_0^3\ln^2 k + 30b_2\beta_0\beta_1^4\beta_2 + 4[b_1 - \kappa(\lambda+\nu)]\beta_0\beta_2^3 - 20b_3\chi\beta_1^2\beta_0^2\ln^2 k - 3\Delta_0\beta_1^2\chi\ln^2 k \\
+20b_2\beta_0^3\beta_2^3 + b_2\beta_1^6 + \alpha\beta_1^2\chi\ln^2 k = 0,
\end{gathered} \tag{374}$$

$$-6b_3\chi\beta_1\beta_0^3\ln^2 k+26b_3\beta_1^3\beta_2\ln^2 k+12[b_1-\kappa(\lambda+\nu)]\beta_0\beta_1\beta_2^2-3\Delta_0\beta_1\beta_0\chi\ln^2 k$$
$$+60b_2\beta_0^2\beta_1^3\beta_2+4[b_1-\kappa(\lambda+\nu)]\beta_1^3\beta_2+6b_2\beta_0\beta_1^5+86b_3\beta_1\beta_2^2\beta_0\ln^2 k+60b_2\beta_0^3\beta_1\beta_2^2=0,$$
$$-\Delta_2\beta_2^2+[b_1-\kappa(\lambda+\nu)]\beta_1^4+32b_3\beta_2^2\beta_0^2\ln^2 k+15b_2\beta_0^2\beta_1^4+6[b_1-\kappa(\lambda+\nu)]\beta_0^2\beta_2^2$$
$$+4\Delta_0\beta_2^2\ln^2 k+12[b_1-\kappa(\lambda+\nu)]\beta_0\beta_1^2\beta_2+60b_2\beta_0^3\beta_1^2\beta_2-4\alpha\beta_2^2\ln^2 k+4b_3\beta_1^4\ln^2 k$$
$$+15b_2\beta_0^4\beta_2^2+56b_3\beta_1^2\beta_0\beta_2\ln^2 k=0,$$
$$-4\alpha\beta_1\beta_2\ln^2 k+4[b_1-\kappa(\lambda+\nu)]\beta_0\beta_1^3+12[b_1-\kappa(\lambda+\nu)]\beta_0^2\beta_1\beta_2+10b_3\beta_1^3\beta_0\ln^2 k$$
$$-2\Delta_2\beta_1\beta_2+38b_3\beta_1\beta_2\beta_0^2\ln^2 k+5\Delta_0\beta_1\beta_2\ln^2 k+30b_2\beta_0^4\beta_1\beta_2+20b_2\beta_0^3\beta_1^3=0,$$
$$-2\Delta_2\beta_0\beta_2+6[b_1-\kappa(\lambda+\nu)]\beta_0^2\beta_1^2+8b_3\beta_2\beta_0^3\ln^2 k+6b_2\beta_0^5\beta_2-\Delta_2\beta_1^2-\alpha\beta_1^2\ln^2 k$$
$$+15b_2\beta_0^4\beta_1^2+8b_3\beta_1^2\beta_0^2\ln^2 k+4[b_1-\kappa(\lambda+\nu)]\beta_0^3\beta_2+\Delta_0\beta_1^2\ln^2 k+4\Delta_0\beta_0\beta_2\ln^2 k=0,$$
$$-2\Delta_2\beta_0\beta_1+4[b_1-\kappa(\lambda+\nu)]\beta_0^3\beta_1+\Delta_0\beta_1\beta_0\ln^2 k+2b_3\beta_1\beta_0^3\ln^2 k+6b_2\beta_0^5\beta_1=0,$$
$$-\Delta_2\beta_0^2+[b_1-\kappa(\lambda+\nu)]\beta_0^4+b_2\beta_0^6=0.$$

On solving the above algebraic Eqs. (374) by using the Maple, one gets the following results:

$$\beta_0=0,\quad \beta_1=0,\quad \beta_2=2\epsilon\sqrt{\frac{6\chi b_3}{b_2}}\ln k,$$
$$\Delta_2=\frac{4\left[-b_2\Delta_0+6b_3[b_1-\kappa(\lambda+\nu)]+96b_3^2\ln^2 k\right]\ln^2 k}{b_2},\tag{375}$$
$$\alpha=\frac{2\left[b_2\Delta_0-3b_3[b_1-\kappa(\lambda+\nu)]-48b_3^2\ln^2 k\right]}{b_2},$$

provided $\chi b_2 b_3>0$ and $\epsilon=\pm1$.

Substituting (375) along with (33) into Eq. (71), one gets the solutions of Eq. (358) in the form:

$$q(x,t)=2\epsilon\sqrt{\frac{6\chi b_3}{b_2}}(\ln k)\left[\frac{4A}{4A^2\exp_k[2(x-ct)]+\chi\exp_k[-2(x-ct)]}\right]e^{i(-\kappa x+\omega t+\theta_0)}.\tag{376}$$

In particular, if we set $\chi=4A^2$ in (376), then we have the bright soliton solution of Eq. (358) as:

$$q(x,t)=2\epsilon\sqrt{\frac{6b_3}{b_2}}(\ln k)\times\operatorname{sech}[2(x-ct)\ln k]e^{i(-\kappa x+\omega t+\theta_0)},\tag{377}$$

while, if $\chi=-4A^2$, one gets the singular soliton solution of Eq. (358) as:

$$q(x,t)=2\epsilon\sqrt{-\frac{6b_3}{b_2}}(\ln k)\times\operatorname{cosech}[2(x-ct)\ln k]e^{i(-\kappa x+\omega t+\theta_0)}.\tag{378}$$

Similarly, we can find many other solutions by choosing other values for p and N.

15. KUDRYASHOV'S LAW

For the Kudryashov's law nonlinearity, we have

$$F(\phi)=\frac{b_1}{\phi^{\frac{n}{2}}}+\frac{b_2}{\phi^n}+b_3\phi^{\frac{n}{2}}+b_4\phi^n,\tag{379}$$

where $b_j\ (j=1,2,3,4)$ give self phase modulation (SPM). Then the nonlinearity index n is the power law parameter.

Equation (1) corresponding to Kudryashov's law nonlinearity (379) is given by:

$$iq_t+iaq_{xxx}+\left(\frac{b_1}{|q|^n}+\frac{b_2}{|q|^{2n}}+b_3|q|^n+b_4|q|^{2n}\right)q$$
$$=\alpha\frac{|q_x|^2}{q^*}+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q\tag{380}$$
$$+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_x q+\nu|q|^{2m}q_x\right],$$

where Eq. (41) reduces to:

$$\Delta_0\phi\phi''-\alpha\phi'^2-\Delta_2\phi^2+b_1\phi^{2-n}+b_2\phi^{2-2n}$$
$$+b_3\phi^{2+n}+b_4\phi^{2+2n}-\kappa(\lambda+\nu)\phi^{2m+2}=0.\tag{381}$$

For integrability, one must select $n=m$. This leads to the modification of Eq. (1) corresponding to Kudryashov's law nonlinearity as:

$$iq_t + iaq_{xxx} + \left(\frac{b_1}{|q|^m} + \frac{b_2}{|q|^{2m}} + b_3|q|^m + b_4|q|^{2m}\right)q$$
$$= \alpha\frac{|q_x|^2}{q^*} + \frac{\beta}{4|q|^2 q^*}\left[2|q|^2\left(|q|^2\right)_{xx} - \left\{\left(|q|^2\right)_x\right\}^2\right] + \gamma q \quad (382)$$
$$+ i\left[\delta q_x + \lambda\left(|q|^{2m}q\right)_x + \mu\left(|q|^{2m}\right)_x q + \nu|q|^{2m}q_x\right].$$

Consequently, Eq. (381) changes to:

$$\Delta_0\phi\phi'' - \alpha\phi'^2 - \Delta_2\phi^2 + b_1\phi^{2-m} + b_2\phi^{2-2m} + b_3\phi^{2+m} + [b_4 - \kappa(\lambda+\nu)]\phi^{2+2m} = 0. \quad (383)$$

Balancing $\phi\phi''$ and ϕ^{2+2m} in Eq. (383), gives the balance number $N = \frac{1}{m}$. Since the balance number is not integer, then we take into consideration the transformation

$$\phi(\xi) = [U(\xi)]^{\frac{1}{m}} \quad (384)$$

where $U(\xi)$ is a new positive function of ξ. Substituting (384) into (383), we have the new equation

$$m\Delta_0 UU'' - \left[\alpha + (m-1)\Delta_0\right]U'^2 + m^2\{b_2 + b_1 U - \Delta_2 U^2 + b_3 U^3 + [b_4 - \kappa(\lambda+\nu)]U^4\} = 0. \quad (385)$$

In the next two subsections, we will solve Eq. (385) using the following two methods:

15.1. New Mapping Method

According to the new mapping method, we balance UU'' with U^4 in Eq. (385) yields the balance number $N = 1$. Now, from (6), the solution of Eq. (385) has the same formal solution (90). Substituting (90) along with (7) into Eq. (385), collecting all the coefficients of $F^l(\xi)[F'(\xi)]^j$ $(l = 0,1,\ldots,8,\ j = 0,1)$ and setting them to zero, we have the following algebraic equations:

$$\begin{gathered}
\frac{4}{3}\alpha s\delta_2^2 - m^2[b_4 - \kappa(\lambda+\nu)]\delta_2^4 - \frac{4}{3}\Delta_0 m\delta_2^2 s - \frac{4}{3}\Delta_0 s\delta_2^2 = 0,\\
-\frac{7}{3}\Delta_0 m\delta_1\delta_2 s + \frac{4}{3}\alpha s\delta_1\delta_2 - 4m^2[b_4 - \kappa(\lambda+\nu)]\delta_1\delta_2^3 - \frac{4}{3}\Delta_0 s\delta_1\delta_2 = 0,\\
-6m^2[b_4 - \kappa(\lambda+\nu)]\delta_1^2\delta_2^2 - 2\Delta_0 h\delta_2^2 - \Delta_0 m\delta_2^2 h - \frac{8}{3}\Delta_0 m\delta_0\delta_2 s - \frac{1}{3}\Delta_0 s\delta_1^2 + \frac{1}{3}\alpha s\delta_1^2\\
-\frac{2}{3}\Delta_0 m\delta_1^2 s - m^2 b_3\delta_2^3 - 4m^2[b_4 - \kappa(\lambda+\nu)]\delta_0\delta_2^3 + 2\alpha h\delta_2^2 = 0,\\
-12m^2[b_4 - \kappa(\lambda+\nu)]\delta_0\delta_1\delta_2^2 - 4m^2[b_4 - \kappa(\lambda+\nu)]\delta_1^3\delta_2 - 2\Delta_0 m\delta_1\delta_2 h - 3m^2 b_3\delta_1\delta_2^2\\
+ 2\alpha h\delta_1\delta_2 - \Delta_0 m\delta_0\delta_1 s - 2\Delta_0 h\delta_1\delta_2 = 0,\\
-4\Delta_0 p\delta_2^2 - 6m^2\left[\delta_0^2\delta_2^2 + 2\delta_0\delta_1^2\delta_2 + \frac{1}{6}\delta_1^4\right][b_4 - \kappa(\lambda+\nu)] - \frac{1}{2}\Delta_0 h\delta_1^2 + \frac{1}{2}\alpha h\delta_1^2\\
+ m^2\Delta_2\delta_2^2 - 3\Delta_0 m\delta_0\delta_2 h - \frac{1}{2}\Delta_0 m\delta_1^2 h - 3m^2 b_3\delta_1^2\delta_2 + 4\alpha p\delta_2^2 - 3m^2 b_3\delta_0\delta_2^2 = 0,\\
-6m^2 b_3\delta_0\delta_1\delta_2 - \Delta_0 m\delta_1\delta_2 p + 2m^2\Delta_2\delta_1\delta_2 - m^2 b_3\delta_1^3 - \Delta_0 m\delta_0\delta_1 h + 4\alpha p\delta_1\delta_2\\
- 4m^2[b_4 - \kappa(\lambda+\nu)]\delta_0\delta_1^3 - 4\Delta_0 p\delta_1\delta_2 - 12m^2[b_4 - \kappa(\lambda+\nu)]\delta_0^2\delta_1\delta_2 = 0,\\
2m^2\Delta_2\delta_0\delta_2 + 4\alpha r\delta_2^2 + \alpha p\delta_1^2 - 4\Delta_0 m\delta_0\delta_2 p - 2m^2\left[2\delta_0^3\delta_2 + 3\delta_0^2\delta_1^2\right][b_4 - \kappa(\lambda+\nu)]\\
+ 2\Delta_0 m\delta_2^2 r - m^2 b_1\delta_2 - 3m^2 b_3\delta_0\delta_1^2 - \Delta_0 p\delta_1^2 + m^2\Delta_2\delta_1^2 - 4\Delta_0 r\delta_2^2 - 3m^2 b_3\delta_0^2\delta_2 = 0,\\
-\Delta_0 m\delta_0\delta_1 p + 2m^2\Delta_2\delta_0\delta_1 - 4m^2[b_4 - \kappa(\lambda+\nu)]\delta_0^3\delta_1 + 2\Delta_0 m\delta_1\delta_2 r + 4\alpha r\delta_1\delta_2\\
-m^2 b_1\delta_1 - 3m^2 b_3\delta_0^2\delta_1 - 4\Delta_0 r\delta_1\delta_2 = 0,\\
-b_2 m^2 - m^2[b_4 - \kappa(\lambda+\nu)]\delta_0^4 + \Delta_0 mr\delta_1^2 - 2\Delta_0 m\delta_0\delta_2 r - m^2 b_3\delta_0^3 + \alpha r\delta_1^2\\
+m^2\Delta_2\delta_0^2 - \Delta_0 r\delta_1^2 - m^2 b_1\delta_0 = 0.
\end{gathered} \quad (386)$$

With the aid of the solutions (8)–(29), we have the following types of solutions:

Type 1. Substituting $s = \frac{3h^2}{16p}$, $r = \frac{16p^2}{27h}$, into the algebraic equations (386) and solve them by Maple, one gets the following results:

$$\delta_0=\frac{30\Delta_2\,(m-1)}{11b_3},\quad \delta_1=0,\quad \delta_2=\frac{60\Delta_2\,(m-1)}{11b_3},\quad p=-\frac{64m^2\Delta_2}{11\Delta_0},\quad h=-\frac{96m^2\Delta_2}{11\Delta_0},$$
$$\alpha=\frac{3}{2}\Delta_0,\quad b_4=\frac{11(2m-1)b_3^2}{200(m-1)^2\Delta_2}+\kappa(\lambda+\nu), \tag{387}$$
$$b_1=\frac{23380\left(m^2-1\right)\Delta_2^2}{3267b_3},\qquad b_2=\frac{12250(m-1)^2\,(2m+1)\,\Delta_2^3}{1089b_3^2},$$

provided

$$b_3\neq 0,\quad \Delta_2\neq 0,\quad \Delta_0\neq 0. \tag{388}$$

If we substitute (387) along with (8)–(12) into Eq. (90), then Eq. (382) has the following solutions.

15.1.1. Soliton solutions.

$$q(x,t)=\left\{\frac{10(m-1)\Delta_2}{33b_3}\left[\frac{27-55\tanh^2\left(8\epsilon m\sqrt{\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}{3+\tanh^2\left(8\epsilon m\sqrt{\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{389}$$

and

$$q(x,t)=\left\{\frac{10(m-1)\Delta_2}{33b_3}\left[\frac{27-55\coth^2\left(8\epsilon m\sqrt{\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}{3+\coth^2\left(8\epsilon m\sqrt{\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{390}$$

provided $b_3\Delta_2\,(m-1)>0$, $\Delta_0\Delta_2>0$ and $\epsilon=\pm1$.

15.1.2. Periodic solutions.

$$q(x,t)=\left\{\frac{10(m-1)\Delta_2}{33b_3}\left[\frac{27+55\tan^2\left(8\epsilon m\sqrt{-\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}{3-\tan^2\left(8\epsilon m\sqrt{-\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{391}$$

and

$$q(x,t)=\left\{\frac{10(m-1)\Delta_2}{33b_3}\left[\frac{27+55\cot^2\left(8\epsilon m\sqrt{-\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}{3-\cot^2\left(8\epsilon m\sqrt{-\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{392}$$

provided $b_3\Delta_2\,(m-1)>0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

Type 2. Substituting $s=\frac{3h^2}{16p}$, $r=0$, into the algebraic Eqs. (386) and solve them by Maple, we get the following results:

$$\delta_0=\frac{30\Delta_2\,(m-1)}{11b_3},\quad \delta_1=0,\quad \delta_2=\frac{60\Delta_2\,(m-1)}{11b_3},\quad p=-\frac{64m^2\Delta_2}{11\Delta_0},\quad h=-\frac{96m^2\Delta_2}{11\Delta_0}\alpha=\frac{3}{2}\Delta_0,$$
$$b_4=\frac{11(2m-1)b_3^2}{200(m-1)^2\Delta_2}+\kappa(\lambda+\nu),\quad b_1=\frac{3900\left(m^2-1\right)\Delta_2^2}{121b_3},\qquad b_2=-\frac{76050(m-1)^2\,(2m+1)\,\Delta_2^3}{1331b_3^2}, \tag{393}$$

provided $b_3\neq 0, \Delta_2\neq 0, \Delta_0\neq 0$.

If we substitute (393) along with (14) and (15) into Eq. (90), then Eq. (382) has the following solutions.

15.1.3. Dark and singular solitons.

$$q(x,t)=\left\{-\frac{10(m-1)\Delta_2}{11b_3}\left[5+8\tanh\left(8\epsilon m\sqrt{-\frac{\Delta_2}{11\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{394}$$

and

$$q(x,t)=\left\{-\frac{10(m-1)\Delta_2}{11b_3}\left[5+8\coth\left(8\epsilon m\sqrt{-\frac{\Delta_2}{11\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{395}$$

respectively, provided $b_3\Delta_2(m-1)<0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

Type 3. Substituting $r=0$, into the algebraic equations (386) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=\frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}},\ \delta_1=0,\ \delta_2=\frac{2}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}},\\ p=-\frac{3m^2\Delta_2}{11\Delta_0},\quad h=\frac{6m^2\Delta_2}{11\Delta_0},\quad s=\frac{4m^2\Delta_2}{11\Delta_0},\quad \alpha=\frac{3}{2}\Delta_0,\\ b_4=\frac{50(4m^2-1)\Delta_2^2}{1089b_2}+\kappa(\lambda+\nu),\quad b_1=\frac{1}{3}(m+1)\Delta_2\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}},\\ b_3=\frac{5(m-1)(2m+1)\Delta_2^2}{1089b_3}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}},\end{gathered} \tag{396}$$

provided

$$(2m+1)\Delta_2b_2<0,\quad b_3\neq0,\quad \Delta_0\neq0. \tag{397}$$

If we substitute (396) along with (16)–(29) into Eq. (90), then Eq. (382) has the following solutions.

15.1.4. Soliton solutions.

$$q(x,t)=\left\{\frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}}\left[1+\frac{18\operatorname{sech}^2\left(\epsilon m\sqrt{-\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)}{9+4\left[1+\tanh\left(\epsilon m\sqrt{-\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{398}$$

$$q(x,t)=\left\{\frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}}\left[1-\frac{18\operatorname{cosech}^2\left(\epsilon m\sqrt{-\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)}{9+4\left[1+\coth\left(\epsilon m\sqrt{-\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{399}$$

provided

$$(2m+1)\Delta_2b_2<0,\quad \Delta_0\Delta_2<0,\quad \epsilon=\pm1. \tag{400}$$

15.1.5. Bright soliton.

$$q(x,t)=\left\{\frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}}\left[1-\frac{12}{5\cosh\left(2\epsilon m\sqrt{-\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)-3}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{401}$$

and

$$q(x,t) = \left\{ \frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}} \left[1 - \frac{6}{5\cosh^2\left(\epsilon m\sqrt{-\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right) - 4} \right] \right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{402}$$

provided

$$(2m+1)\Delta_2 b_2 < 0, \quad \Delta_0\Delta_2 < 0, \quad \epsilon = \pm 1. \tag{403}$$

15.1.6. Singular soliton.

$$q(x,t) = \left\{ \frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}} \left[1 + \frac{6}{5\sinh^2\left(\epsilon m\sqrt{-\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right) + 4} \right] \right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{404}$$

provided $(2m+1)\Delta_2 b_2 < 0$, $\Delta_0\Delta_2 < 0$, $\epsilon = \pm 1$.

15.1.7. Periodic solutions.

$$q(x,t) = \left\{ \frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}} \left[1 + \frac{6\sec^2\left(\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)}{3 + 4\tan\left(\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)} \right] \right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{405}$$

$$q(x,t) = \left\{ \frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}} \left[1 + \frac{6\operatorname{cosec}^2\left(\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)}{3 + 4\cot\left(\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)} \right] \right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{406}$$

$$q(x,t) = \left\{ \frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}} \left[1 - \frac{6\sec^2\left(\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)}{5 - 4\sec^2\left(\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)} \right] \right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{407}$$

$$q(x,t) = \left\{ \frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}} \left[1 - \frac{6\operatorname{cosec}^2\left(\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)}{5 - 4\operatorname{cosec}^2\left(\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)} \right] \right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{408}$$

$$q(x,t) = \left\{ \frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}} \left[1 - \frac{12\sec\left(2\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)}{5 - 3\sec\left(2\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)} \right] \right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{409}$$

$$q(x,t) = \left\{ \frac{1}{5}\sqrt{-\frac{33b_2}{(2m+1)\Delta_2}} \left[1 - \frac{12\operatorname{cosec}\left(2\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)}{5 - 3\operatorname{cosec}\left(2\epsilon m\sqrt{\frac{3\Delta_2}{11\Delta_0}}(x-ct)\right)} \right] \right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{410}$$

provided

$$(2m+1)\Delta_2 b_2 < 0, \quad \Delta_0\Delta_2 < 0, \quad \epsilon = \pm 1. \tag{411}$$

15.2. Addedum to Kudryashov's Method

According to this method, we balance UU'' with U^4 in Eq. (385), one gets the relation:

$$2N + 2p = 4N \Rightarrow N = p. \tag{412}$$

Now, we will discuss the following cases:

Case 1. If we choose $p = 1$, then $N = 1$. Thus, we deduce that from (31) that Eq. (385) has the same formal solution (113). Substituting (113) along with (32) into Eq. (385), collecting all the coefficients of each power of $[R(\xi)]^{m_1}$ $[R'(\xi)]^j$, $(m_1 = 0,1,2,\ldots,4, \quad j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
-[b_4-\kappa(\lambda+\nu)]m^2\beta_1^4+\beta_1^2\left(\ln^2 k\right)\Delta_0\chi-\beta_1^2\left(\ln^2 k\right)\alpha\chi+\beta_1^2\left(\ln^2 k\right)\Delta_0 m\chi=0,\\
-4[b_4-\kappa(\lambda+\nu)]m^2\beta_0\beta_1^3+2\Delta_0\beta_1\left(\ln^2 k\right)m\beta_0\chi-m^2b_3\beta_1^3=0,\\
-\beta_1^2\left(\ln^2 k\right)\Delta_0+m^2\Delta_2\beta_1^2-3m^2b_3\beta_0\beta_1^2+\beta_1^2\left(\ln^2 k\right)\alpha-6[b_4-\kappa(\lambda+\nu)]m^2\beta_0^2\beta_1^2=0,\\
2m^2\Delta_2\beta_0\beta_1-3m^2b_3\beta_0^2\beta_1-m^2b_1\beta_1-4[b_4-\kappa(\lambda+\nu)]m^2\beta_0^3\beta_1-\Delta_0\beta_1\left(\ln^2 k\right)m\beta_0=0,\\
-m^2b_1\beta_0-m^2b_3\beta_0^3-m^2b_2+m^2\Delta_2\beta_0^2-m^2[b_4-\kappa(\lambda+\nu)]\beta_0^4=0.
\end{gathered} \tag{413}$$

On solving the above algebraic Eqs. (413) by using the Maple, one gets the following results:

$$\begin{gathered}
\beta_0=\frac{\ln k}{4m}\sqrt{\frac{2(2m-1)\Delta_0\chi}{b_4-\kappa(\lambda+\nu)}},\quad \beta_1=\frac{\ln k}{2m}\sqrt{\frac{2(2m-1)\Delta_0\chi}{b_4-\kappa(\lambda+\nu)}},\\
\alpha=\frac{3}{2}\Delta_0,\quad \Delta_2=\frac{(3\chi-2)\Delta_0\ln^2 k}{4m^2},\quad b_1=-\frac{(m+1)(\chi-2)\Delta_0\ln^3 k}{8m^3}\sqrt{\frac{2(2m-1)\Delta_0\chi}{b_4-\kappa(\lambda+\nu)}}\\
b_2=-\frac{(4m^2-1)(\chi-4)\chi\Delta_0^2\ln^4 k}{64m^4[b_4-\kappa(\lambda+\nu)]},\quad b_3=-\frac{(m-1)[b_4-\kappa(\lambda+\nu)]\ln k}{m(2m-1)}\sqrt{\frac{2(2m-1)\Delta_0\chi}{b_4-\kappa(\lambda+\nu)}},
\end{gathered} \tag{414}$$

provided $(2m-1)\Delta_0\chi[b_4-\kappa(\lambda+\nu)]>0$.

Substituting (414) along with (33) into Eq. (113), one gets the solutions of Eq. (382) in the form:

$$q(x,t)=\left\{\frac{\ln k}{4m}\sqrt{\frac{2(2m-1)\Delta_0\chi}{b_4-\kappa(\lambda+\nu)}}\left[1+\frac{8A}{4A^2\exp_k[(x-ct)]+\chi\exp_k[-(x-ct)]}\right]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{415}$$

provided $[b_3-\kappa(\lambda+\nu)]\chi\Delta_0<0$.

In particular, if we set $\chi = 4A^2$ in (415), then we have the bright soliton solution of Eq. (382) as:

$$q(x,t)=\left\{\frac{\ln k}{2m}\sqrt{\frac{2(2m-1)\Delta_0}{b_4-\kappa(\lambda+\nu)}}[A+\operatorname{sech}[(x-ct)\ln k]]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{416}$$

while, if we set $\chi = -4A^2$ in (415), then we have the singular soliton solution of Eq. (382) as:

$$q(x,t)=\left\{\frac{\ln k}{2m}\sqrt{-\frac{2(2m-1)\Delta_0}{b_4-\kappa(\lambda+\nu)}}[A+\operatorname{cosech}[(x-ct)\ln k]]\right\}^{\frac{1}{m}}e^{i(-\kappa x+\omega t+\theta_0)}. \tag{417}$$

Case 2. If we choose $p = 2$, then $N = 2$. Thus, we deduce that from (31) that Eq. (131) has the same formal solution (119). Substituting (119) along with Eq. (32) into Eq. (385), collecting all the coefficients of each power of $[R(\xi)]^{m_2}$ $[R'(\xi)]^j$, $(m_2 = 0,1,2,\ldots,8, \ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
4\left(\ln^2 k\right)\Delta_0\chi\beta_2^2 - 4\left(\ln^2 k\right)\alpha\chi\beta_2^2 - [b_4 - \kappa(\lambda+\nu)]m^2\beta_2^4 + 4\left(\ln^2 k\right)\Delta_0 m\beta_2^2\chi = 0, \\
4\left(\ln^2 k\right)\Delta_0\chi\beta_1\beta_2 + 7\Delta_0\beta_1\left(\ln^2 k\right)m\beta_2\chi - 4[b_4 - \kappa(\lambda+\nu)]m^2\beta_1\beta_2^3 - 4\left(\ln^2 k\right)\alpha\chi\beta_1\beta_2 = 0, \\
-4[b_4 - \kappa(\lambda+\nu)]m^2\beta_0\beta_2^3 - \beta_1^2\left(\ln^2 k\right)\alpha\chi + \beta_1^2\left(\ln^2 k\right)\Delta_0\chi - m^2 b_3\beta_2^3 + 2\beta_1^2\left(\ln^2 k\right)\Delta_0 m\chi \\
+ 8\left(\ln^2 k\right)\Delta_0 m\beta_0\beta_2\chi - 6[b_4 - \kappa(\lambda+\nu)]m^2\beta_1^2\beta_2^2 = 0, \\
-12[b_4 - \kappa(\lambda+\nu)]m^2\beta_0\beta_1\beta_2^2 - 3m^2 b_3\beta_1\beta_2^2 + 3\Delta_0\beta_1\left(\ln^2 k\right)m\beta_0\chi - 4[b_4 - \kappa(\lambda+\nu)]m^2\beta_1^3\beta_2 = 0, \\
-12[b_4 - \kappa(\lambda+\nu)]m^2\beta_0\beta_1^2\beta_2 - 4\left(\ln^2 k\right)\Delta_0\beta_2^2 - 3m^2 b_3\beta_1^2\beta_2 + m^2\Delta_2\beta_2^2 - [b_4 - \kappa(\lambda+\nu)]m^2\beta_1^4 \\
- 3m^2 b_3\beta_0\beta_2^2 + 4\left(\ln^2 k\right)\alpha\beta_2^2 - 6[b_4 - \kappa(\lambda+\nu)]m^2\beta_0^2\beta_2^2 = 0, \\
-4\Delta_0\beta_1\left(\ln^2 k\right)\beta_2 + 2m^2\Delta_2\beta_1\beta_2 - \Delta_0\beta_1\left(\ln^2 k\right)m\beta_2 - 12[b_4 - \kappa(\lambda+\nu)]m^2\beta_0^2\beta_1\beta_2 \\
- 6m^2 b_3\beta_0\beta_1\beta_2 - m^2 b_3\beta_1^3 - 4[b_4 - \kappa(\lambda+\nu)]m^2\beta_0\beta_1^3 + 4\left(\ln^2 k\right)\alpha\beta_1\beta_2 = 0, \\
-6[b_4 - \kappa(\lambda+\nu)]m^2\beta_0^2\beta_1^2 + 2m^2\Delta_2\beta_0\beta_2 + m^2\Delta_2\beta_1^2 - \beta_1^2\left(\ln^2 k\right)\Delta_0 - m^2 b_1\beta_2 - 3m^2 b_3\beta_0\beta_1^2 \\
-4\left(\ln^2 k\right)\Delta_0 m\beta_0\beta_2 - 4[b_4 - \kappa(\lambda+\nu)]m^2\beta_0^3\beta_2 + \beta_1^2\left(\ln^2 k\right)\alpha - 3m^2 b_3\beta_0^2\beta_2 = 0, \\
2m^2\Delta_2\beta_0\beta_1 - 3m^2 b_3\beta_0^2\beta_1 - m^2 b_1\beta_1 - 4[b_4 - \kappa(\lambda+\nu)]m^2\beta_0^3\beta_1 - \Delta_0\beta_1\left(\ln^2 k\right)m\beta_0 = 0, \\
m^2\Delta_2\beta_0^2 - m^2 b_1\beta_0 - m^2 b_2 - m^2[b_4 - \kappa(\lambda+\nu)]\beta_0^4 - m^2 b_3\beta_0^3 = 0.
\end{gathered} \tag{418}$$

On solving the above algebraic Eqs. (418) by using the Maple, one gets the following results:

$$\begin{gathered}
\beta_0 = \frac{\ln k}{2m}\sqrt{\frac{2(2m-1)\Delta_0\chi}{b_4 - \kappa(\lambda+\nu)}},\quad \beta_1 = \frac{\ln k}{m}\sqrt{\frac{2(2m-1)\Delta_0\chi}{b_4 - \kappa(\lambda+\nu)}},\quad \alpha = \frac{3}{2}\Delta_0,\quad \Delta_2 = \frac{(3\chi-2)\Delta_0\ln^2 k}{m^2}, \\
b_1 = \frac{(m+1)(\chi-2)\Delta_0\ln^3 k}{m^3}\sqrt{\frac{2(2m-1)\Delta_0\chi}{b_4 - \kappa(\lambda+\nu)}},\quad b_2 = -\frac{(4m^2-1)(\chi-4)\chi\Delta_0^2\ln^4 k}{4m^4[b_4 - \kappa(\lambda+\nu)]}, \\
b_3 = -\frac{2(m-1)[b_4 - \kappa(\lambda+\nu)]\ln k}{m(2m-1)}\sqrt{\frac{2(2m-1)\Delta_0\chi}{b_4 - \kappa(\lambda+\nu)}},
\end{gathered} \tag{419}$$

provided $(2m-1)\Delta_0\chi[b_4 - \kappa(\lambda+\nu)] > 0$.

Substituting (419) along with (33) into Eq. (113), one gets the solutions of Eq. (382) in the form:

$$q(x,t) = \left\{\frac{\ln k}{2m}\sqrt{\frac{2(2m-1)\Delta_0\chi}{b_4 - \kappa(\lambda+\nu)}}\left[1 + \frac{8A}{4A^2\exp_k[2(x-ct)] + \chi\exp_k[-2(x-ct)]}\right]\right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{420}$$

provided $[b_3 - \kappa(\lambda+\nu)]\chi\Delta_0 < 0$.

In particular, if we set $\chi = 4A^2$ in (420), then we have the bright soliton solution of Eq. (382) as:

$$q(x,t) = \left\{\frac{\ln k}{m}\sqrt{\frac{2(2m-1)\Delta_0}{b_4 - \kappa(\lambda+\nu)}}[A + \operatorname{sech}[2(x-ct)\ln k]]\right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{421}$$

while, if we set $\chi = -4A^2$ in (420), then we have the singular soliton solution of Eq. (382) as:

$$q(x,t) = \left\{\frac{\ln k}{m}\sqrt{-\frac{2(2m-1)\Delta_0}{b_4 - \kappa(\lambda+\nu)}}[A + \operatorname{cosech}[2(x-ct)\ln k]]\right\}^{\frac{1}{m}} e^{i(-\kappa x+\omega t+\theta_0)}. \tag{422}$$

Similarly, we can find many other solutions by choosing other values for p and N.

16. GENERALIZED KUDRYASHOV'S LAW

For the generalized Kudryashov's law nonlinearity, we have

$$F(\phi)=\frac{b_1}{\phi^{2n}}+\frac{b_2}{\phi^{\frac{3n}{2}}}+\frac{b_3}{\phi^{n}}+\frac{b_4}{\phi^{\frac{n}{2}}}+b_5\phi^{\frac{n}{2}}+b_6\phi^{n}+b_7\phi^{\frac{3n}{2}}+b_8\phi^{2n}, \tag{423}$$

where $b_j(j=1,2,\ldots,8)$ give self phase modulation (SPM). Then the nonlinearity index n is the power law parameter.

Equation (1) corresponding to generalized Kudryashov's law nonlinearity (423) is given by:

$$\begin{aligned} &iq_t+iaq_{xxx}+\left(\frac{b_1}{|q|^{4n}}+\frac{b_2}{|q|^{3n}}+\frac{b_3}{|q|^{2n}}+\frac{b_4}{|q|^{n}}+b_5|q|^{n}+b_6|q|^{2n}+b_7|q|^{3n}+b_8|q|^{4n}\right)q\\ &=\alpha\frac{|q_x|^2}{q^*}+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_xq+\nu|q|^{2m}q_x\right],\end{aligned} \tag{424}$$

where Eq. (41) reduces to:

$$\begin{aligned}&\Delta_0\phi\phi''-\alpha\phi'^2-\Delta_2\phi^2+b_1\phi^{2-4n}+b_2\varphi^{2-3n}+b_3\phi^{2-2n}+b_4\phi^{2-n}+b_5\phi^{2+n}+b_6\phi^{2+2n}\\ &\quad+b_7\phi^{2+3n}+b_8\phi^{2+4n}-\kappa(\lambda+\nu)\phi^{2m+2}=0.\end{aligned} \tag{425}$$

For integrability, one must select $n=m$. This leads to the modification of Eq. (1) corresponding to generalized Kudryashov's law nonlinearity as:

$$\begin{aligned} &iq_t+iaq_{xxx}+\left(\frac{b_1}{|q|^{4m}}+\frac{b_2}{|q|^{3m}}+\frac{b_3}{|q|^{2m}}+\frac{b_4}{|q|^{m}}+b_5|q|^{m}+b_6|q|^{2m}+b_7|q|^{3m}+b_8|q|^{4m}\right)q\\ &=\alpha\frac{|q_x|^2}{q^*}+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_xq+\nu|q|^{2m}q_x\right],\end{aligned} \tag{426}$$

Consequently, Eq. (425) changes to:

$$\begin{aligned}&\Delta_0\phi\phi''-\alpha\phi'^2-\Delta_2\phi^2+b_1\phi^{2-4m}\\ &+b_2\phi^{2-3m}+b_3\phi^{2-2m}+b_4\phi^{2-m}+b_5\phi^{2+m}\\ &+[b_6-\kappa(\lambda+\nu)]\phi^{2+2m}+b_7\phi^{2+3m}+b_8\phi^{2+4m}=0.\end{aligned} \tag{427}$$

Balancing $\phi\phi''$ and ϕ^{2+4m} in Eq. (427), gives the balance number $N=\frac{1}{2m}$. Since the balance number is not integer, then we take into consideration the transformation

$$\phi(\xi)=[U(\xi)]^{\frac{1}{2m}}, \tag{428}$$

where $U(\xi)$ is a new positive function of ξ. Substituting (428) into (427), we have the new equation

$$\begin{aligned}&2m\Delta_0UU''-[\alpha+(2m-1)\Delta_0]U'^2\\ &+4m^2\{b_1+b_3U-\Delta_2U^2\\ &+[b_6-\kappa(\lambda+\nu)]U^3+b_8U^4\}\\ &+4m^2b_2U^{\frac{1}{2}}+4m^2b_4U^{\frac{3}{2}}+4m^2b_5U^{\frac{5}{2}}+4m^2b_7U^{\frac{7}{2}}=0.\end{aligned} \tag{429}$$

For integrability, one must select

$$b_2=0,\quad b_4=0,\quad b_5=0,\quad b_7=0. \tag{430}$$

This leads to the modification of Eq. (1) corresponding to generalized Kudryashov's law nonlinearity as:

$$\begin{aligned}&iq_t+iaq_{xxx}+\left(\frac{b_1}{|q|^{4m}}+\frac{b_3}{|q|^{2m}}+b_6|q|^{2m}+b_8|q|^{4m}\right)q\\ &=\alpha\frac{|q_x|^2}{q^*}+\frac{\beta}{4|q|^2q^*}\left[2|q|^2\left(|q|^2\right)_{xx}-\left\{\left(|q|^2\right)_x\right\}^2\right]+\gamma q\\ &+i\left[\delta q_x+\lambda\left(|q|^{2m}q\right)_x+\mu\left(|q|^{2m}\right)_xq+\nu|q|^{2m}q_x\right].\end{aligned} \tag{431}$$

Consequently, Eq. (429) changes to

$$\begin{aligned}&2m\Delta_0UU''-[\alpha+(2m-1)\Delta_0]U'^2\\ &+4m^2\{b_1+b_3U-\Delta_2U^2\\ &+[b_6-\kappa(\lambda+\nu)]U^3+b_8U^4\}=0.\end{aligned} \tag{432}$$

In the next two subsections, we will solve Eq. (432) using the following two methods.

16.1. New Mapping Method

According to the new mapping method, we balance UU'' with U^4 in Eq. (432) yields the balance number $N=1$. Now, from (6), the solution of Eq. (432) has the same formal solution (90). Substituting (90) along with (7) into Eq. (432), collecting all the coefficients of $F^l(\xi)\left[F'(\xi)\right]^j$ $(l=0,1,\ldots,8,\ j=0,1)$ and setting them to zero, we have the following algebraic equations:

$$\frac{4}{3}\Delta_0 s\delta_2^2-\frac{4}{3}\alpha s\delta_2^2+4b_8m^2\delta_2^4+\frac{8}{3}\Delta_0 ms\delta_2^2=0,$$
$$\frac{4}{3}\Delta_0 s\delta_1\delta_2-\frac{4}{3}\alpha s\delta_1\delta_2+16b_8m^2\delta_1\delta_2^3+\frac{14}{3}\Delta_0 ms\delta_1\delta_2=0,$$
$$\frac{16}{3}\Delta_0 m\delta_0\delta_2 s-2\alpha h\delta_2^2-\frac{1}{3}\alpha s\delta_1^2+2\Delta_0 h\delta_2^2+\frac{1}{3}\Delta_0 s\delta_1^2+24b_8m^2\delta_1^2\delta_2^2+\frac{4}{3}\Delta_0 ms\delta_1^2$$
$$+2\Delta_0 mh\delta_2^2+4[b_6-\kappa(\lambda+\nu)]m^2\delta_2^3+16b_8m^2\delta_0\delta_2^3=0,$$
$$12[b_6-\kappa(\lambda+\nu)]m^2\delta_1\delta_2^2+48b_8m^2\delta_0\delta_1\delta_2^2-2\alpha h\delta_1\delta_2+4\Delta_0 mh\delta_1\delta_2$$
$$+2\Delta_0 m\delta_0\delta_1 s+16b_8m^2\delta_1^3\delta_2+2\Delta_0 h\delta_1\delta_2=0,$$
$$4\Delta_0 p\delta_2^2+\Delta_0 mh\delta_1^2-4\Delta_2m^2\delta_2^2+6\Delta_0 m\delta_0\delta_2 h+12[b_6-\kappa(\lambda+\nu)]m^2\left[\delta_1^2\delta_2+\delta_0\delta_2^2\right]$$
$$+24b_8m^2\delta_0^2\delta_2^2-\frac{1}{2}\alpha h\delta_1^2+48b_8m^2\delta_0\delta_1^2\delta_2+4b_8m^2\delta_1^4+\frac{1}{2}\Delta_0 h\delta_1^2-4\alpha p\delta_2^2=0,$$
$$4[b_6-\kappa(\lambda+\nu)]m^2\delta_1^3-8\Delta_2m^2\delta_1\delta_2+2\Delta_0 m\delta_0\delta_1 h+2\Delta_0 mp\delta_1\delta_2-4\alpha p\delta_1\delta_2 \tag{433}$$
$$+24[b_6-\kappa(\lambda+\nu)]m^2\delta_0\delta_1\delta_2+48b_8m^2\delta_0^2\delta_1\delta_2+4\Delta_0 p\delta_1\delta_2+16b_8m^2\delta_0\delta_1^3=0,$$
$$4\Delta_0 r\delta_2^2+8\Delta_0 m\delta_0\delta_2 p-8\Delta_2m^2\delta_0\delta_2+12[b_6-\kappa(\lambda+\nu)]m^2\delta_0\delta_1^2+24b_8m^2\delta_0^2\delta_1^2$$
$$-\alpha p\delta_1^2+16b_8m^2\delta_0^3\delta_2-4\alpha r\delta_2^2+\Delta_0 p\delta_1^2-4\Delta_2m^2\delta_1^2-4\Delta_0 mr\delta_2^2$$
$$+12[b_6-\kappa(\lambda+\nu)]m^2\delta_0^2\delta_2+4b_3m^2\delta_2=0,$$
$$-8\Delta_2m^2\delta_0\delta_1+16b_8m^2\delta_0^3\delta_1+4b_3m^2\delta_1+12[b_6-\kappa(\lambda+\nu)]m^2\delta_0^2\delta_1-4\alpha r\delta_1\delta_2$$
$$+4\Delta_0 r\delta_1\delta_2-4\Delta_0 mr\delta_1\delta_2+2\Delta_0 m\delta_0\delta_1 p=0,$$
$$-4\Delta_2m^2\delta_0^2+4b_3m^2\delta_0+\Delta_0 r\delta_1^2+4[b_6-\kappa(\lambda+\nu)]m^2\delta_0^3+4\Delta_0 m\delta_0\delta_2 r$$
$$-2\Delta_0 mr\delta_1^2+4b_1m^2+4b_8m^2\delta_0^4-\alpha r\delta_1^2=0.$$

With the aid of the solutions (8)–(29), we have the following types of solutions:

Type 1. Substituting $s=\frac{3h^2}{16p}$, $r=\frac{16p^2}{27h}$, into the algebraic Eqs. (433) and solve them by Maple, one gets the following results:

$$\delta_0=\frac{30(2m-1)\Delta_2}{11[b_6-\kappa(\lambda+\nu)]},\quad \delta_1=0,\quad \delta_2=\frac{60(2m-1)\Delta_2}{11[b_6-\kappa(\lambda+\nu)]},\quad p=-\frac{256m^2\Delta_2}{11\Delta_0},$$
$$h=-\frac{384m^2\Delta_2}{11\Delta_0},\quad \alpha=\frac{3}{2}\Delta_0,\quad b_1=\frac{12250\left(16m^3-12m^2+1\right)\Delta_2^3}{1089[b_6-\kappa(\lambda+\nu)]^2}, \tag{434}$$
$$b_3=\frac{23380\left(4m^2-1\right)\Delta_2^2}{3267[b_6-\kappa(\lambda+\nu)]},\quad b_8=\frac{11(4m-1)[b_6-\kappa(\lambda+\nu)]^2}{200(2m-1)^2\Delta_2},$$

provided $\Delta_0\neq 0, b_6-\kappa(\lambda+\nu)\neq 0$ and $\Delta_2\neq 0$.

If we substitute (434) along with (8)–(12) into Eq. (90), then Eq. (431) has the following solutions.

16.1.1. Soliton solutions.

$$q(x,t)=\left\{\frac{30(2m-1)\Delta_2}{11[b_6-\kappa(\lambda+\nu)]}\left[1-\frac{64\tanh^2\left(16\epsilon m\sqrt{\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}{27+3\tanh^2\left(16\epsilon m\sqrt{\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{435}$$

and

$$q(x,t)=\left\{\frac{30(2m-1)\Delta_2}{11[b_6-\kappa(\lambda+\nu)]}\left[1-\frac{64\coth^2\left(16\epsilon m\sqrt{\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}{27+3\coth^2\left(16\epsilon m\sqrt{\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{436}$$

provided $(2m-1)[b_6-\kappa(\lambda+\nu)]\Delta_2>0, \Delta_0\Delta_2>0$ and $\epsilon=\pm1$.

16.1.2. Periodic solutions.

$$q(x,t)=\left\{\frac{30(2m-1)\Delta_2}{11[b_6-\kappa(\lambda+\nu)]}\left[1+\frac{64\tan^2\left(16\epsilon m\sqrt{-\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}{27-3\tan^2\left(16\epsilon m\sqrt{-\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{437}$$

and

$$q(x,t)=\left\{\frac{30(2m-1)\Delta_2}{11[b_6-\kappa(\lambda+\nu)]}\left[1+\frac{64\cot^2\left(16\epsilon m\sqrt{-\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}{27-3\cot^2\left(16\epsilon m\sqrt{-\frac{\Delta_2}{33\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{438}$$

provided $(2m-1)[b_6-\kappa(\lambda+\nu)]\Delta_2>0, \Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

Type 2. Substituting $s=\frac{3h^2}{16p}$, $r=0$, into the algebraic Eqs. (433) and solve them by Maple, we get the following results:

$$\begin{gathered}\delta_0=\frac{30(2m-1)\Delta_2}{11[b_6-\kappa(\lambda+\nu)]},\quad \delta_1=0,\quad \delta_2=\frac{60(2m-1)\Delta_2}{11[b_6-\kappa(\lambda+\nu)]},\quad p=-\frac{256m^2\Delta_2}{11\Delta_0},\\ h=-\frac{384m^2\Delta_2}{11\Delta_0},\quad \alpha=\frac{3}{2}\Delta_0,\quad b_1=-\frac{76050\left(16m^3-12m^2+1\right)\Delta_2^3}{1331[b_6-\kappa(\lambda+\nu)]^2},\\ b_3=\frac{3900\left(4m^2-1\right)\Delta_2^2}{121[b_6-\kappa(\lambda+\nu)]},\quad b_8=\frac{11(4m-1)[b_6-\kappa(\lambda+\nu)]^2}{200(2m-1)^2\Delta_2},\end{gathered} \tag{439}$$

provided $\Delta_0\neq0, b_6\neq0$, $\Delta_2\neq0$ and $\epsilon=\pm1$.

If we substitute (439) along with (14) and (15) into Eq. (90), then Eq. (431) has the following solutions.

16.1.3. Dark and singular solitons.

$$q(x,t)=\left\{-\frac{30(2m-1)\Delta_2}{33[b_6-\kappa(\lambda+\nu)]}\left[5+8\tanh\left(16\epsilon m\sqrt{-\frac{\Delta_2}{11\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{440}$$

and

$$q(x,t)=\left\{-\frac{30(2m-1)\Delta_2}{33[b_6-\kappa(\lambda+\nu)]}\left[5+8\coth\left(16\epsilon m\sqrt{-\frac{\Delta_2}{11\Delta_0}}(x-ct)\right)\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{441}$$

respectively, provided $(4m-1)[b_6-\kappa(\lambda+\nu)]\Delta_2<0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

Type 3. Substituting $r=0$, into the algebraic Eqs. (433) and solve them by Maple, we get the following results:

$$\delta_0=-\frac{(2m-1)\Delta_2}{33[b_6-\kappa(\lambda+\nu)]},\quad \delta_1=0,\quad \delta_2=-\frac{2(2m-1)\Delta_2}{33[b_6-\kappa(\lambda+\nu)]},\quad p=-\frac{12m^2\Delta_2}{11\Delta_0},$$

$$h=\frac{24m^2\Delta_2}{11\Delta_0},\quad s=\frac{16m^2\Delta_2}{11\Delta_0},\quad \alpha=\frac{3}{2}\Delta_0,\quad b_1=-\frac{25\left(16m^3-12m^2+1\right)\Delta_2^3}{35937\left[b_6-\kappa(\lambda+\nu)\right]^2},\tag{442}$$

$$b_3=-\frac{5\left(4m^2-1\right)\Delta_2^2}{99\left[b_6-\kappa(\lambda+\nu)\right]},\quad b_8=-\frac{66(4m-1)\left[b_6-\kappa(\lambda+\nu)\right]^2}{(2m-1)^2\Delta_2},$$

provided $\Delta_0\neq 0, b_6-\kappa(\lambda+\nu)\neq 0$ and $\Delta_2\neq 0$.

If we substitute (442) along with (16)–(29) into Eq. (90), then Eq. (431) has the following solutions.

16.1.4. Soliton solutions.

$$q(x,t)=\left\{-\frac{(2m-1)\Delta_2}{33\left[b_6-\kappa(\lambda+\nu)\right]}\left[1+\frac{18\operatorname{sech}^2\left(\epsilon m\sqrt{-\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}{9+4\left[1+\tanh\left(\epsilon m\sqrt{-\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{443}$$

and

$$q(x,t)=\left\{-\frac{(2m-1)\Delta_2}{33\left[b_6-\kappa(\lambda+\nu)\right]}\left[1-\frac{18\operatorname{cosech}^2\left(\epsilon m\sqrt{-\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}{9+4\left[1+\coth\left(\epsilon m\sqrt{-\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)\right]^2}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{444}$$

provided $(2m-1)\left[b_6-\kappa(\lambda+\nu)\right]\Delta_2<0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

16.1.5. Bright soliton.

$$q(x,t)=\left\{-\frac{(2m-1)\Delta_2}{33\left[b_6-\kappa(\lambda+\nu)\right]}\left[1+\frac{12}{3-5\cosh\left(2\epsilon m\sqrt{-\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{445}$$

and

$$q(x,t)=\left\{-\frac{(2m-1)\Delta_2}{33\left[b_6-\kappa(\lambda+\nu)\right]}\left[1+\frac{18}{4-5\cosh^2\left(\epsilon m\sqrt{-\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{446}$$

provided $(2m-1)\left[b_6-\kappa(\lambda+\nu)\right]\Delta_2<0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

16.1.6. Singular soliton.

$$q(x,t)=\left\{-\frac{(2m-1)\Delta_2}{33\left[b_6-\kappa(\lambda+\nu)\right]}\left[1+\frac{18}{4+5\sinh^2\left(\epsilon m\sqrt{-\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)},\tag{447}$$

provided $(2m-1)\left[b_6-\kappa(\lambda+\nu)\right]\Delta_2<0$, $\Delta_0\Delta_2<0$ and $\epsilon=\pm1$.

16.1.7. Periodic solutions.

$$q(x,t)=\left\{-\frac{(2m-1)\Delta_2}{33[b_6-\kappa(\lambda+\nu)]}\left[1+\frac{6\sec^2\left(\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}{3+4\tan\left(\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{448}$$

$$q(x,t)=\left\{-\frac{(2m-1)\Delta_2}{33[b_6-\kappa(\lambda+\nu)]}\left[1+\frac{6\operatorname{cosec}^2\left(\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}{3+4\cot\left(\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{449}$$

$$q(x,t)=\left\{-\frac{(2m-1)\Delta_2}{33[b_6-\kappa(\lambda+\nu)]}\left[1-\frac{6\sec^2\left(\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}{5-4\sec^2\left(\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{450}$$

$$q(x,t)=\left\{-\frac{(2m-1)\Delta_2}{33[b_6-\kappa(\lambda+\nu)]}\left[1-\frac{6\operatorname{cosec}^2\left(\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}{5-4\operatorname{cosec}^2\left(\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{451}$$

$$q(x,t)=\left\{-\frac{(2m-1)\Delta_2}{33[b_6-\kappa(\lambda+\nu)]}\left[1-\frac{12\sec\left(2\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}{5-3\sec\left(2\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{452}$$

$$q(x,t)=\left\{-\frac{(2m-1)\Delta_2}{33[b_6-\kappa(\lambda+\nu)]}\left[1-\frac{12\operatorname{cosec}\left(2\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}{5-3\operatorname{cosec}\left(2\epsilon m\sqrt{\frac{12\Delta_2}{11\Delta_0}}(x-ct)\right)}\right]\right\}^{\frac{1}{2m}}e^{i(-\kappa x+\omega t+\theta_0)}, \tag{453}$$

provided $(2m-1)[b_6-\kappa(\lambda+\nu)]\Delta_2<0,\quad \Delta_0\Delta_2>0$ and $\epsilon=\pm1$.

16.2. Addendum to Kudryashov's Method

According to this method, we balance UU'' with U^4 in Eq. (432), one gets the relation:

$$2N+2p=4N\Rightarrow N=p. \tag{454}$$

Now, we will discuss the following cases:

Case 1. If we choose $p=1$, then $N=1$. Thus, we deduce that from (31) that Eq. (432) has the same formal solution (113). Substituting (113) along with (32) into Eq. (432), collecting all the coefficients of each power of $[R(\xi)]^{m_1}\,[R'(\xi)]^j$, $(m_1=0,1,2,\ldots,4,\ j=0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
\beta_1^2\left(\ln^2 k\right)\alpha\chi-2\Delta_0\beta_1^2\left(\ln^2 k\right)m\chi\\
+4m^2b_8\beta_1^4-\beta_1^2\left(\ln^2 k\right)\Delta_0\chi=0,\\
-4\Delta_0\beta_1\left(\ln^2 k\right)m\beta_0\chi+16m^2b_8\beta_0\beta_1^3\\
+4m^2[b_6-\kappa(\lambda+\nu)]\beta_1^3=0,\\
24m^2b_8\beta_0^2\beta_1^2-4m^2\Delta_2\beta_1^2\\
+12m^2[b_6-\kappa(\lambda+\nu)]\beta_0\beta_1^2\\
-\beta_1^2\left(\ln^2 k\right)\alpha+\beta_1^2\left(\ln^2 k\right)\Delta_0=0,\\
2\Delta_0\beta_1\left(\ln^2 k\right)m\beta_0+16m^2b_8\beta_0^3\beta_1-8m^2\Delta_2\beta_0\beta_1\\
+4m^2b_3\beta_1+12m^2[b_6-\kappa(\lambda+\nu)]\beta_0^2\beta_1=0,\\
4m^2[b_6-\kappa(\lambda+\nu)]\beta_0^3+4m^2b_8\beta_0^4\\
+4m^2b_3\beta_0-4m^2\Delta_2\beta_0^2+4m^2b_1=0.
\end{gathered} \tag{455}$$

On solving the above algebraic Eqs. (455) by using the Maple, one gets the following results:

$$\beta_0 = -\frac{(2m-1)\chi\Delta_0 \ln^2 k}{8m^2[b_6 - \kappa(\lambda+\nu)]}, \quad \beta_1 = -\frac{(2m-1)\chi\Delta_0 \ln^2 k}{4m^2[b_6 - \kappa(\lambda+\nu)]},$$
$$\alpha = \frac{3}{2}\Delta_0, \quad \Delta_2 = \frac{(3\chi-2)\Delta_0 \ln^2 k}{16m^2}, \quad b_1 = -\frac{1}{2}(4m+1)(\chi-4)\Delta_0\left[\frac{(2m-1)\chi\Delta_0 \ln^3 k}{32m^3[b_6-\kappa(\lambda+\nu)]}\right]^2, \tag{456}$$
$$b_3 = -\frac{(4m^2-1)(\chi-2)\chi\Delta_0^2 \ln^4 k}{64m^4[b_6-\kappa(\lambda+\nu)]}, \qquad b_8 = \frac{2(4m-1)m^2[b_6-\kappa(\lambda+\nu)]^2}{(2m-1)^2\chi\Delta_0 \ln^2 k},$$

provided $\chi\Delta_0 \neq 0$ and $b_6 - \kappa(\lambda+\nu) \neq 0$.

Substituting (456) along with (33) into Eq. (113), one gets the solutions of Eq. (431) in the form:

$$q(x,t) = \left\{-\frac{(2m-1)\chi\Delta_0 \ln^2 k}{8m^2[b_6-\kappa(\lambda+\nu)]}\left[1 + \frac{8A}{4A^2 \exp_k[(x-ct)] + \chi \exp_k[-(x-ct)]}\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{457}$$

provided $[b_6 - \kappa(\lambda+\nu)]\chi\Delta_0 < 0$.

In particular, if we set $\chi = 4A^2$ in (457), then we have the bright soliton solution of Eq. (431) as:

$$q(x,t) = \left\{-\frac{(2m-1)A\Delta_0 \ln^2 k}{8m^2[b_6-\kappa(\lambda+\nu)]}[A + \operatorname{sech}[(x-ct)\ln k]]\right\}^{\frac{1}{2m}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{458}$$

while, if we set $\chi = -4A^2$ in (457), then we have the singular soliton solution of Eq. (431) as:

$$q(x,t) = \left\{\frac{(2m-1)A\Delta_0 \ln^2 k}{8m^2[b_6-\kappa(\lambda+\nu)]}[A + \operatorname{cosech}[(x-ct)\ln k]]\right\}^{\frac{1}{2m}} e^{i(-\kappa x + \omega t + \theta_0)}. \tag{459}$$

Case 2. If we choose $p = 2$, then $N = 2$. Thus, we deduce that from (31) that Eq. (432) has the same formal solution (119). Substituting (119) along with Eq. (32) into Eq. (432), collecting all the coefficients of each power of $[R(\xi)]^{m_2}$ $[R'(\xi)]^j$, $(m_2 = 0,1,2,\ldots,8,\ j = 0,1)$ and setting each of these coefficients to zero, one gets the following set of algebraic equations:

$$\begin{gathered}
4m^2 b_8\beta_2^4 + 4(\ln^2 k)\alpha\chi\beta_2^2 - 8\Delta_0(\ln^2 k)m\beta_2^2\chi - 4(\ln^2 k)\Delta_0\chi\beta_2^2 = 0,\\
16m^2 b_8\beta_1\beta_2^3 + 4(\ln^2 k)\alpha\chi\beta_1\beta_2 - 4(\ln^2 k)\Delta_0\chi\beta_1\beta_2 - 14\Delta_0(\ln^2 k)m\beta_1\beta_2\chi = 0,\\
-\beta_1^2(\ln^2 k)\Delta_0\chi - 4\Delta_0\beta_1^2(\ln^2 k)m\chi + 16m^2 b_8\beta_0\beta_2^3 + 24m^2 b_8\beta_1^2\beta_2^2\\
+ 4m^2[b_6-\kappa(\lambda+\nu)]\beta_2^3 - 16\Delta_0(\ln^2 k)m\beta_0\beta_2\chi + \beta_1^2(\ln^2 k)\alpha\chi = 0,\\
-6\Delta_0\beta_1(\ln^2 k)m\beta_0\chi + 12m^2[b_6-\kappa(\lambda+\nu)]\beta_1\beta_2^2 + 16m^2 b_8\beta_1^3\beta_2 + 48m^2 b_8\beta_0\beta_1\beta_2^2 = 0,\\
12m^2[b_6-\kappa(\lambda+\nu)]\beta_1^2\beta_2 + 4(\ln^2 k)\Delta_0\beta_2^2 + 4m^2 b_8\beta_1^4 - 4(\ln^2 k)\alpha\beta_2^2\\
+12m^2[b_6-\kappa(\lambda+\nu)]\beta_0\beta_2^2 + 24m^2 b_8\beta_0^2\beta_2^2 - 4m^2\Delta_2\beta_2^2 + 48m^2 b_8\beta_0\beta_1^2\beta_2 = 0,\\
2\Delta_0(\ln^2 k)m\beta_1\beta_2 - 8m^2\Delta_2\beta_1\beta_2 + 16m^2 b_8\beta_0\beta_1^3 + 4m^2[b_6-\kappa(\lambda+\nu)][6\beta_0\beta_1\beta_2 + \beta_1^3]\\
+ 4(\ln^2 k)\Delta_0\beta_1\beta_2 - 4(\ln^2 k)\alpha\beta_1\beta_2 + 48m^2 b_8\beta_0^2\beta_1\beta_2 = 0,\\
\beta_1^2(\ln^2 k)\Delta_0 - 8m^2\Delta_2\beta_0\beta_2 - \beta_1^2(\ln^2 k)\alpha + 4m^2 b_3\beta_2 + 16m^2 b_8\beta_0^3\beta_2 - 4m^2\Delta_2\beta_1^2\\
+24m^2 b_8\beta_0^2\beta_1^2 + 12m^2[b_6-\kappa(\lambda+\nu)]\beta_0\beta_1^2 + 12m^2[b_6-\kappa(\lambda+\nu)]\beta_0^2\beta_2 + 8\Delta_0(\ln^2 k)m\beta_0\beta_2 = 0,\\
+2\Delta_0\beta_1(\ln^2 k)m\beta_0 + 16m^2 b_8\beta_0^3\beta_1 - 8m^2\Delta_2\beta_0\beta_1 + 4m^2 b_3\beta_1 + 12m^2[b_6-\kappa(\lambda+\nu)]\beta_0^2\beta_1 = 0,\\
-4m^2\Delta_2\beta_0^2 + 4m^2[b_6-\kappa(\lambda+\nu)]\beta_0^3 + 4m^2 b_8\beta_0^4 + 4m^2 b_3\beta_0 + 4m^2 b_1 = 0.
\end{gathered} \tag{460}$$

On solving the above algebraic Eqs. (460) by using the Maple, one gets the following results:

$$\beta_0 = -\frac{(2m-1)\chi\Delta_0 \ln^2 k}{2m^2[b_6 - \kappa(\lambda+\nu)]}, \quad \beta_1 = 0, \quad \beta_2 = -\frac{(2m-1)\chi\Delta_0 \ln^2 k}{m^2[b_6 - \kappa(\lambda+\nu)]},$$

$$\alpha = \frac{3}{2}\Delta_0, \quad \Delta_2 = \frac{(3\chi-2)\Delta_0 \ln^2 k}{4m^2}, \quad b_1 = -\frac{1}{2}(4m+1)(\chi-4)\Delta_0\left[\frac{(2m-1)\chi\Delta_0 \ln^3 k}{4m^3[b_6 - \kappa(\lambda+\nu)]}\right]^2, \tag{461}$$

$$b_3 = -\frac{(4m^2-1)(\chi-2)\chi\Delta_0^2 \ln^4 k}{4m^4[b_6 - \kappa(\lambda+\nu)]}, \quad b_8 = \frac{(4m-1)m^2[b_6 - \kappa(\lambda+\nu)]^2}{2(2m-1)^2\chi\Delta_0 \ln^2 k},$$

provided $\chi\Delta_0 \neq 0$ and $b_6 \neq 0$.

Substituting (461) along with (33) into Eq. (113), one gets the solutions of Eq. (431) in the form:

$$q(x,t) = \left\{-\frac{(2m-1)\chi\Delta_0 \ln^2 k}{2m^2[b_6 - \kappa(\lambda+\nu)]}\left[1 + \frac{8A}{4A^2 \exp_k[2(x-ct)] + \chi \exp_k[-2(x-ct)]}\right]\right\}^{\frac{1}{2m}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{462}$$

provided $[b_3 - \kappa(\lambda+\nu)]\chi\Delta_0 < 0$.

In particular, if we set $\chi = 4A^2$ in (462), then we have the bright soliton solution of Eq. (431) as:

$$q(x,t) = \left\{-\frac{2(2m-1)A\Delta_0 \ln^2 k}{m^2[b_6 - \kappa(\lambda+\nu)]}[A + \operatorname{sech}[2(x-ct)\ln k]]\right\}^{\frac{1}{2m}} e^{i(-\kappa x + \omega t + \theta_0)}, \tag{463}$$

while, if we set $\chi = -4A^2$ in (462), then we have the singular soliton solution of Eq. (431) as:

$$q(x,t) = \left\{\frac{2(2m-1)A\Delta_0 \ln^2 k}{m^2[b_6 - \kappa(\lambda+\nu)]}[A + \operatorname{cosech}[2(x-ct)\ln k]]\right\}^{\frac{1}{2m}} e^{i(-\kappa x + \omega t + \theta_0)}. \tag{464}$$

Similarly, we can find many other solutions by choosing other values for p and N.

17. CONCLUSIONS

This paper contains innovative results on PC solitons that stem from CGLE studied with a dozen forms of nonlinear refractive index. Such forms of SPM led to the emergence of bright, dark and singular optical solitons. These solitons are thus a true asset in physics and telecommunications engineering. They lead to several follow–up studies that can be conducted immediately. One immediate area to expand is the retrieval of conservation laws. These conservation laws for the twelve forms of SPM with PC solitons will surely be a gateway to enhance the study of such solitons with CGLE. These con laws would enable the study of soliton perturbation theory, collision induced timing jitter and stochastic perturbation with the retrieval of mean free velocity of thesolitons by formulating the corresponding Langevin equations. Later, additional schemes would yield the dynamics of temporal evolution of soliton parameters. These are variational principle, collective variables and moment method. Other features that must be addressed are from quasi–stationary solitons, optical couplers, magneto–optic waveguides, dispersive solitons and many others [32–40]. Another avenue to venture would be to consider differential group delay followed by the extension to DWDM topology and dispersion–flattened fibers. One more issue is to study the PC solitons with CGLE as the governing model in the context of Bragg gratings, in case of low CD, with these dozen forms of SPM. Such studies are all under way and they would be gradually and surely reported with time. The inquisitive readers are suggested to just hang in there!

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