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# Optical vortices in waveguides with spatial dependence of the nonlinear refractive index

Valeri Slavchev[1,2] · Ivan Bozhikoliev[1] · Zhelyazko Zamanchev[3] · Aneliya Dakova[1,3] · Kamen Kovachev[1] · Anjan Biswas[4,5,6,7,8]

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**Abstract**
In the present work, the formation of optical vortex in waveguides, with spatial dependence of the nonlinear refractive index, is studied. The propagation of such type of laser pulses is governed by a system of amplitude equations for $x$ and $y$ components of the electrical field in which the effects of second-order dispersion and self-phase modulation are taken into account. The corresponding system of equations is solved analytically. New class of exact solutions, describing the generation of vortex structures in the optical fibers with spatial dependence of the nonlinear refractive index and anomalous dispersion, are found. These optical vortices admit only amplitude type singularities. Their stability is a result of the delicate balance between diffraction and nonlinearity, as well as nonlinearity and angular distribution. This kind of singularities can be observed as a depolarization of the vector field in the laser spot.

**Keywords** Vector amplitude equation · Optical vortices · Amplitude singularities

## 1 Introduction

The classical optical vortices are referred to beams that have singularities in the phases. These structures are solutions of two-dimensional paraxial scalar equation of Leontovich (Nye and Berry 1974; Soskin et al. 1997). They are usually created outside the laser cavity by using optical holograms and different optical masks (Rozas 1999; Aksenov et al. 2018; Hansinger et al. 2016; Coullet et al. 1989; Heckenberg et al. 1992; Brunet et al. 2010) and admit angular dependence of the electrical field or helical phase distribution.

On the other hand, new vector type of optical vortices with singularities in the components of the amplitude of the electrical field were recently found (Bozhikoliev et al.

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✉ Valeri Slavchev
valeri.slavchev@mu-plovdiv.bg

Extended author information available on the last page of the article

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2019). The behavior of optical vortices in different waveguides is described by the nonlinear amplitude equation in which it is included a term, corresponding to the spatial dependence of the nonlinear refractive index $n_2(x^2 + y^2)$. Amplitude modulations in such optical structures are observed in the case of studying the vector form of the electrical field and they are investigated in the frames of a system of two scalar nonlinear amplitude equations for the $x$ and $y$ components of the vector electric field. A solution of the 3D+1 nonlinear Schrodinger equation for optical fibers with spatial dependence of the nonlinear refractive index was found for the first time in a vector form by the authors in Dakova et al. (2007) and Kovachev et al. (2004).

In our previous work, vortex structures with spatial dependence of the linear refractive index in gradient inhomogeneous waveguides were studied (Dakova et al. 2019). A new class of vortex solutions for optical fibers with a concave refractive index profile has been found. Their stability is due not only to the balance between diffraction and nonlinearity, but also to the nonlinearity and angular distribution. The spatial dependence of the linear refractive index leads to the formation of optical vortices in the field of the intensity components of the laser pulse. This naturally raises the question: *is it possible such type of vortex structure to exist in a medium with a spatial dependence of the nonlinear refractive index*?

The main goal of present work is to find analytical vortex solutions of the vector nonlinear amplitude equation for optical fibers with quadratic nonlinear refractive index.

In recent decades, many authors (Wang et al. 2018; Zhang et al. 2020; Porfirev et al. 2021; Fatkhiev 2021) have reported a significant progress on the generation of optical vortices. Their applications in active resonators have been demonstrated in Maguid (2018), Uren et al. (2019) and Sroor et al. (2020). The dynamics of the vortices during their propagation in optical fibers has been practically investigated by the authors in Kotlyar et al. (1998), Bolshtyansky et al. (1999), Karpeev and Khonina (2007) and Khonina et al. (2010). The behavior of these structures in a gradient fiber is observed by authors in Slavchev et al. (2020), Dakova et al. (2018) and Slavchev et al. (2021). Helical structures of the vortex solutions for the components of the electical field are found in Ng et al. (2010). Optical vortices have a number of applications in the field of high resolution microscopy, optical tweezers, quantum information transfer, optical vortex trapping and many others (Rui et al. 2015; Gahagan and Swartzlander 1996; Datta and Saha 2020).

## 2 Basic equation

The equation describing the propagation of optical vortices in waveguides with spatial dependence of the nonlinear refractive index in the vector form Dakova et al. (2007) and Kovachev et al. (2004) is:

$$-i\alpha \frac{\partial \vec{A}}{\partial z} + \frac{|\beta|}{2} \frac{\partial^2 \vec{A}}{\partial t^2} + \frac{1}{2} \Delta_\perp \vec{A} + \gamma (x^2 + y^2) |\vec{A}|^2 \vec{A} = 0 \tag{1}$$

where $\vec{A}$ is the vector amplitude function of the pulse envelope, $t$ is time, $\alpha$, $\beta$ and $\gamma$ are constants, characterizing respectively the number of oscillations under the pulse's envelope, dispersion and nonlinearity of the fiber. Here, $\Delta_\perp$ is the transverse operator of Laplace. The operator and the constants are of the kind:

$$\alpha = k_0 z_0, \mid \beta \mid = k_0 u^2 \mid k'' \mid, \Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \gamma = \frac{\alpha^2}{2} n_2 \mid A_0 \mid^2, \tag{2}$$

where $k_0$ is the wave number, $u$, $k''$, $n_2$ are the group velocity, the second-order of linear dispersion and the nonlinear refractive index of the medium, $z_0 = ut_0$ is the initial longitudinal length of the pulse ($t_0$ is time duration of the pulse) and $A_0$ is the magnitude of the initial amplitude of the pulse. We have in mind that $\vec{A} = (A_x, A_y, 0)$. It is accepted that the axis $O_z$ coincides with the geometrical axis of the fiber. Thus, it is convenient to work in cylindrical coordinates:

$$\begin{aligned} &x = r\cos\theta, y = r\sin\theta, \theta = \arctan(y/x), r^2 = x^2 + y^2, \\ &\Delta_\perp = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \end{aligned} \tag{3}$$

After a couple of transformations, the scalar equations describing the evolution of the components $A_x$ and $A_y$ of the vector function $\vec{A}$, written in polar coordinates, can be presented as follows:

$$\begin{aligned} &- i\alpha\frac{\partial A_x}{\partial z} + \frac{\mid \beta \mid}{2}\frac{\partial^2 A_x}{\partial t^2} + \frac{1}{2}\left(\frac{1}{r}\frac{\partial A_x}{\partial r} + \frac{\partial^2 A_x}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A_x}{\partial \theta^2}\right) + \gamma r^2 \mid A_x^2 + A_y^2 \mid A_x = 0, \\ &- i\alpha\frac{\partial A_y}{\partial z} + \frac{\mid \beta \mid}{2}\frac{\partial^2 A_y}{\partial t^2} + \frac{1}{2}\left(\frac{1}{r}\frac{\partial A_y}{\partial r} + \frac{\partial^2 A_y}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A_y}{\partial \theta^2}\right) + \gamma r^2 \mid A_x^2 + A_y^2 \mid A_y = 0. \end{aligned} \tag{4}$$

## 3 Mathematical method

In order to find the solutions for the components $A_x$ and $A_y$ of the vector amplitude function $\vec{A}$ of the pulse, the following substitutions in the system of Eqs. (4) are made:

$$\begin{aligned} &A_x(r,\theta,z,t) = P_x(r,\theta)e^{i(az+bt)}, \\ &A_y(r,\theta,z,t) = P_y(r,\theta)e^{i(az+bt)}, \end{aligned} \tag{5}$$

where $a$ and $b$ are constants about to be defined, $P_x$ and $P_y$ are new unknown real functions. After several transformations we obtain:

$$\begin{aligned} &(2\alpha a + b^2 \mid \beta \mid) = \frac{1}{P_x}\left(\frac{1}{r}\frac{\partial P_x}{\partial r} + \frac{\partial^2 P_x}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 P_x}{\partial \theta^2}\right) + 2\gamma r^2 \mid P_x^2 + P_y^2 \mid, \\ &(2\alpha a + b^2 \mid \beta \mid) = \frac{1}{P_y}\left(\frac{1}{r}\frac{\partial P_y}{\partial r} + \frac{\partial^2 P_y}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 P_x}{\partial \theta^2}\right) + 2\gamma r^2 \mid P_x^2 + P_y^2 \mid . \end{aligned} \tag{6}$$

The left sides of the equations above are the same constant expressions. Their right sides are functions of the variables $r$ and $\theta$. In order to fulfill the equalities, we assume that:

$$2\alpha a + b^2 \mid \beta \mid = 0. \tag{7}$$

From this equality we find a connection between the constants $a$ and $b$:

$$a = -\frac{|\beta|}{2\alpha} b^2. \tag{8}$$

Having in mind the expression (7), Eqs. (6) take the form:

$$\begin{aligned}
&\frac{1}{r}\frac{\partial P_x}{\partial r} + \frac{\partial^2 P_x}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 P_x}{\partial \theta^2} + 2\gamma r^2 \mid P_x^2 + P_y^2 \mid P_x = 0,\\
&\frac{1}{r}\frac{\partial P_y}{\partial r} + \frac{\partial^2 P_y}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 P_y}{\partial \theta^2} + 2\gamma r^2 \mid P_x^2 + P_y^2 \mid P_y = 0.
\end{aligned} \tag{9}$$

We make another pair of substitutions as:

$$\begin{aligned}
P_x(r,\theta) &= R_x(r)e^{in\theta},\\
P_y(r,\theta) &= R_y(r)e^{in\theta},
\end{aligned} \tag{10}$$

where $n = const$. By using the expressions (10), the system of Eqs. (9) can be presented in the form:

$$\begin{aligned}
&\frac{1}{r}\frac{\partial R_x}{\partial r} + \frac{\partial^2 R_x}{\partial r^2} - \frac{n^2}{r^2}R_x + 2\gamma r^2 (R_x^2 + R_y^2) R_x = 0,\\
&\frac{1}{r}\frac{\partial R_y}{\partial r} + \frac{\partial^2 R_y}{\partial r^2} - \frac{n^2}{r^2}R_y + 2\gamma r^2 (R_x^2 + R_y^2) R_y = 0.
\end{aligned} \tag{11}$$

Taking into account the nonlinear terms in Eqs. (11), it is convenient to search for solutions of the kind:

$$\begin{aligned}
R_x &= B r^{\mu} \cos(m r^{\eta}),\\
R_y &= B r^{\mu} \sin(m r^{\eta}).
\end{aligned} \tag{12}$$

where $B, m, \mu, \eta$ are constants, about to be defined.

By substituting expressions (12) in the system of differential equations (11) and after short transformations, we obtain:

$$\begin{aligned}
&r^{\mu-2}\cos(mr^{\eta})\left[\mu^2 - n^2 - m^2\eta^2 r^{2\eta} + 2\gamma B^2 r^{2\mu+4}\right] - m\eta r^{\mu+\eta-2}\sin(mr^{\eta})[2\mu+\eta] = 0,\\
&r^{\mu-2}\sin(mr^{\eta})\left[\mu^2 - n^2 - m^2\eta^2 r^{2\eta} + 2\gamma B^2 r^{2\mu+4}\right] - m\eta r^{\mu+\eta-2}\cos(mr^{\eta})[2\mu+\eta] = 0.
\end{aligned} \tag{13}$$

To fulfill the equalities in (13), it is needed, that the coefficients in front of the respective trigonometric functions in both equations to be equal to zero. In this way, we obtain the following system of two algebraic equations:

$$\begin{aligned}
&2\mu + \eta = 0,\\
&\mu^2 - n^2 - m^2\eta^2 r^{2\eta} + 2\gamma B^2 r^{2\mu+4} = 0.
\end{aligned} \tag{14}$$

By using the system of equations above we can define that:

$$\mu = -\frac{2}{3}, \eta = \frac{4}{3}, n^2 = \mu^2, B = \frac{4m}{3\sqrt{2\gamma}}. \tag{15}$$

Thus, the following exact analytical solutions for the functions $P_x$ and $P_y$, describing the optical vortices, propagating in fibers with spatial dependece of the nonlinear refractive index are found:

$$
\begin{aligned}
P_x &= \frac{4m}{3\sqrt{2\gamma}} r^{-\frac{2}{3}} \cos\left(mr^{\frac{4}{3}}\right) e^{i\frac{2}{3}\theta}, \\
P_y &= \frac{4m}{3\sqrt{2\gamma}} r^{-\frac{2}{3}} \sin\left(mr^{\frac{4}{3}}\right) e^{i\frac{2}{3}\theta}.
\end{aligned}
\tag{16}
$$

As a next step, going back through all the substitutions and assumptions made by now, the solutions for the components $A_x$ and $A_y$ of the vector amplitude function $\vec{A}$ of the optical vortex, satisfying the basic equations (1), can be presented in the form:

$$
\begin{aligned}
A_x &= \frac{4m}{3\sqrt{2\gamma}} r^{-\frac{2}{3}} \cos\left(mr^{\frac{4}{3}}\right) e^{i\frac{2}{3}\theta - i(az+bt)}, \\
A_y &= \frac{4m}{3\sqrt{2\gamma}} r^{-\frac{2}{3}} \sin\left(mr^{\frac{4}{3}}\right) e^{i\frac{2}{3}\theta - i(az+bt)},
\end{aligned}
\tag{17}
$$

where $a = -\frac{|\beta|}{2\alpha} b^2$ and $m$ is an arbitrary real number.

## 4 Graphics of vortex solutions

In Fig. 1a, b the intensity profiles of the vortex structures for the $x$ and $y$ components of the vector $\vec{A}$ are presented. The maxima in the intensity of the $A_x$ component coincide with the minima in $A_y$. As a result the ring structures in the field of the total pulse intensity are not observed and in the intensity profile $|\vec{A}|^2$ vortices are not found (Fig. 1c) due to the compensation of the rotation in the two components $A_x$ and $A_y$.

The rotation of the vector $\vec{A}$ in the center of the optical vortices is shown in Fig. 2. The vector diagrams of this type of amplitude vortex structures are characterized by depolarization in the spot of the laser pulse.

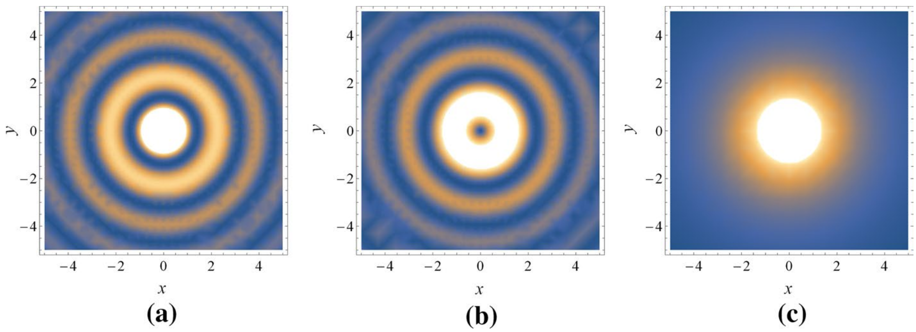


**Fig. 1** Intensity profiles of the components **a** $A_x$, **b** $A_y$, presented by Eq. (17) and **c** the total intensity profile for $m = 1$. The maxima in the intensity of the $A_x$ component coincide with the minima in $A_y$. That is way, the ring structures in the field of the total pulse intensity are not observed

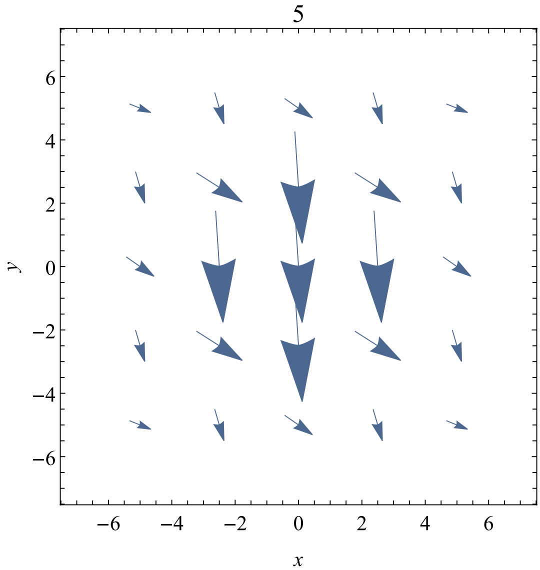


**Fig. 2** Diagram of the vector amplitude function for $m = 1$. Significant rotation of the vector $\vec{A}$ in the center of the vortices is observed. The direction of the vector field has different values at different points of the pulse spot (depolarization)

On Fig. 3 the intensity profiles for the components $x$ and $y$ of the vector $\vec{A}$ in the case of higher value of the vortex parameter $m$ are presented. It is shown that the vortex solutions admit more rich internal structures. The significant growth in the number of rings leads to a narrowing of their minima and maxima. This trend is intensified by the increase of the vortex parameter $m$. As it can be seen, for a higher value of $m$ (in the case of $m = 4$) a tangible change in vorticity and depolarization in the vector diagram (Fig. 4) is observed. Comparing this result with the vector diagram for $m = 1$ we can conclude that for $m = 4$ the level of depolarization is significantly greater.

The obtained results present two different possibilities for the generation of vortex structures in the field of the intensity profile of the components of the vector amplitude function: by filtering one of the components—linear polarization or by observing depolarization in the vector diagram in the spot of the optical pulse.

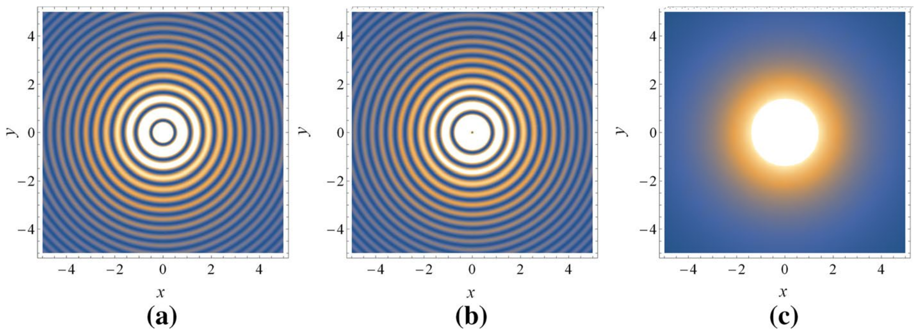


**Fig. 3** Intensity profiles of the components **a** $A_x$, **b** $A_y$, represented by Eq. (17) and **c** the total intensity profile for $m = 4$. The intensity profiles of the components admit more rich internal structures. The significant growth in the number of rings leads to a narrowing of their minima and maxima

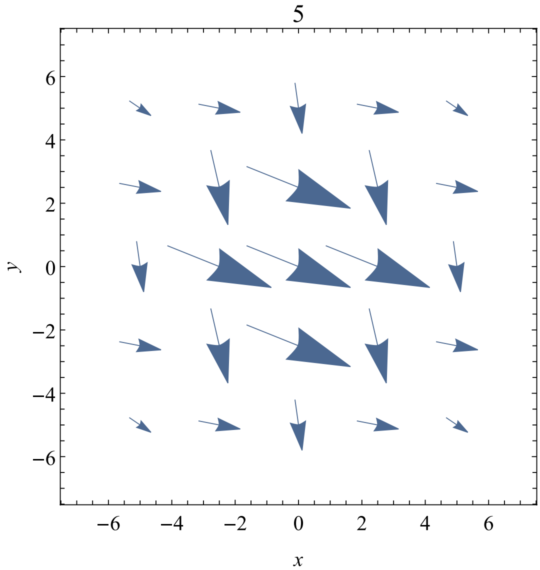


**Fig. 4** Diagram of the vector amplitude function for $m = 4$. Comparing this result with the vector diagram for $m = 1$ it can be concluded that for $m = 4$ the level of depolarization is significantly greater

## 5 Conclusion

In the present work vortex solutions for the components $A_x$ and $A_y$ of the vector amplitude function $\vec{A}$ are found. The graphics of the obtained solutions for different values of the vortex parameters $m$ are presented. If we look at the intensity profiles of the components (Figs. 1, 3), the maxima in the $A_x$ coincide with the minima in $A_y$. As a result, the ring structures in the field of the total pulse intensity are not observed, due to the compensation of the rotation in the two components. The value of the parameter $m$ determines the number of rings observed in the profiles of the intensity components of the vector amplitude function. The significant growth in the number of rings leads to a narrowing of their minima and maxima. This trend is intensified by the increase of the vortex parameter $m$. Depolarization in the diagram of the vector field is observed. Each point of the spot of the optical pulse has a different orientation of the field.

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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## Authors and Affiliations

**Valeri Slavchev[1,2] · Ivan Bozhikoliev[1] · Zhelyazko Zamanchev[3] · Aneliya Dakova[1,3] · Kamen Kovachev[1] · Anjan Biswas[4,5,6,7,8]**

1 Nonlinear and Fiber Optics, Institute of Electronics, Bulgarian Academy of Science, 72 Tzarigradcko Shossee, 1784 Sofia, Bulgaria

2 Department of Medical Physics and Biophysics, Medical University-Plovdiv, 15 Vasil Aprilov Bul., 4002 Plovdiv, Bulgaria

3 Physics and Technology Faculty, University of Plovdiv “Paisii Hilendarski”, 24 Tsar Asen Str., 4000 Plovdiv, Bulgaria

4 Department of Applied Mathematics, National Research Nuclear University, 31 Kashirskoe Hwy, Moscow, Russian Federation 115409

5 Mathematical Modeling and Applied Computation (MMAC) Research Group, Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia

6 Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa-0204, Pretoria, South Africa

7 Department of Physics, Chemistry and Mathematics, Alabama A &M University, Normal, AL 35762–4900, USA

8 Department of Applied Sciences, Cross-Border Faculty, Dunarea de Jos University of Galat, 111 Domneasca Street, 800201 Galati, Romania