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Optical Solitons in Fiber Bragg Gratings with Polynomial Law Nonlinearity and Cubic–Quartic Dispersive Reflectivity

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Abstract—Optical solitons with fiber Bragg gratings and polynomial law of nonlinear refractive index are addressed in the paper. The auxiliary equation approach together with an addendum to Kudryashov's method identify soliton solutions to the model. Singular periodic solutions emerge from these integration schemes as a byproduct.

Keywords: solitons, cubic–quartic, Bragg gratings

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1. INTRODUCTION

The two sources of stable soliton transmission across intercontinental distances are chromatic dispersion (CD) and self–phase modulation (SPM). Occasionally, it is this CD that can run low. There are a variety of approaches that can compensate for this crisis situation. This paper utilizes two approaches. One is the consideration of cubic–quartic (CQ) dispersive effects that stem from third-order dispersion (3OD) and fourth-order dispersion (4OD) which replaces CD. The second source is Bragg gratings that introduces dispersive reflectivity which compensates for the low count of CD [1–10]. The current model is being studied with polynomial law of nonlinearity as opposed to Kerr law. The three nonlinear terms, that constitute polynomial law of nonlinear refractive index, is the source of SPM for the current work. This restructured form of soliton transmission dynamics through fiber optic cables, having Bragg gratings, is the model of study in this paper.

Two integration approaches identify the soliton solutions to the model that travels down such an engineered optical fiber. These are auxiliary equation approach and an addendum to Kudryashov's approach [11–20]. These two schemes collectively yield bright, dark and singular soliton solutions to the model. The existence criteria for such solitons, with regards to the respective parameter restrictions, are also enumerated in the paper. These integration algorithms are illustrated and are successfully implemented to address the models. The details are exhibited in the rest of the paper after an intro to the newly engineered model.

1.1. Governing Model

The dimensionless form of the coupled cubic-quartic perturbed NLSE in fiber Bragg gratings with polynomial law nonlinearity is written in the form:

$$\begin{aligned} &iq_t + ia_1 r_{xxx} + b_1 r_{xxxx} + \left(c_1|q|^2 + d_1|r|^2\right)q \\ &+ \left(\xi_1|q|^4 + \eta_1|q|^2|r|^2 + \zeta_1|r|^4\right)q \\ &+\left(l_1|q|^6 + m_1|q|^4|r|^2 + n_1|q|^2|r|^4 + p_1|r|^6\right)q \\ &+ i\alpha_1 q_x + \beta_1 r + \sigma_1 q^* r^2 \\ &= i\left[\gamma_1\left(|q|^2 q\right)_x + \theta_1\left(|q|^2\right)_x q + \mu_1|q|^2 q_x\right], \end{aligned} \tag{1}$$

and

$$\begin{aligned} & ir_t + ia_2q_{xxx} + b_2q_{xxxx} + \left(c_2|r|^2 + d_2|q|^2\right)r \\ & + \left(\xi_2|r|^4 + \eta_2|r|^2|q|^2 + \zeta_2|q|^4\right)r \\ & + \left(l_2|r|^6 + m_2|r|^4|q|^2 + n_2|r|^2|q|^4 + p_2|q|^6\right)r \\ & + i\alpha_2r_x + \beta_2q + \sigma_2r^*q^2 \\ & = i\left[\gamma_2\left(|r|^2r\right)_x + \theta_2\left(|r|^2\right)_x r + \mu_2|r|^2r_x\right], \end{aligned} \tag{2}$$

where $q(x,t)$ and $r(x,t)$ represent forward and backward propagation wave profile, respectively, while a_j $(j = 1, 2)$ are 3OD and b_j are 4OD as long as $i = \sqrt{-1}$. Then, c_j, ξ_j and l_j are the coefficients of self-phase modulation (SPM) terms; d_j, η_j, ζ_j, m_j, n_j and p_j represent the cross-phase modulation (XPM) terms; α_j represent the inter-modal dispersion and next β_j are the detuning parameters. Also, σ_j give the four wave-mixing (4WM) effect, while γ_j represent the self-steepening to avoid the shock waves formulation. Finally, θ_j and μ_j stem from nonlinear dispersion coefficients.

Equations (1) and (2) represent the coupled-mode equations for Bragg gratings with polynomial law of nonlinearity as opposed to the usual form of Kerr law of nonlinearity where all fibers are manufactured with such a nonlinear form of refractive index. This polynomial law of nonlinearity is a dual extension to the Kerr type of nonlinearity and is occasionally, otherwise, referred to as cubic–quintic–septic nonlinear form. These additional terms are taken into consideration when higher order approximations are included. Next, the CD is replaced with CQ dispersive effects that comprise of two sources of dispersion as opposed to a single source. This is just to compensate for low count of CD. In Bragg gratings, it is the dispersive reflectivity, with 3OD and 4OD, that comes into play. Now, the perturbation terms that stem from self-steepening and nonlinear dispersion and inter-modal dispersion are also included. These are treated as strong perturbation terms without stepping into beyond slowly varying envelope approximation [11]. The effect of Raman scattering is tacitly excluded since it is non-Hamiltonian and is rendered to be non-integrable [11]. Such is the governing model that will be handled in this work, from an integrability standpoint, which will reveal soliton solutions. It must be noted that such a model is not yet handled experimentally and is therefore at a pre-lab stage. Thus, at a theoretical stage, a range of mathematical properties are being investigated into. Additional features such as retrieval of conservation laws, studying the model with fractional temporal evolution are on the horizon and form the tip of the iceberg.

This article is organized as follows: In Section 2, the mathematical preliminaries are introduced. In Sections 3 and 4, the new auxiliary equation method and the addendum to Kudryashov's method are applied to retrieve and exhibit soliton solutions to the model. In Section 5, a few conclusive statements are worded.

2. MATHEMATICAL PRELIMINARIES

In order to solve Eqs. (1) and (2), we assume:

$$\begin{aligned} q(x,t) &= \phi_1(\xi)\exp[i\psi(x,t)], \\ r(x,t) &= \phi_2(\xi)\exp[i\psi(x,t)], \end{aligned} \tag{3}$$

such that

$$\xi = x - vt, \quad \psi(x,t) = -\kappa x + \omega t + \theta_0, \tag{4}$$

where v, κ, ω and θ_0 are all nonzero constants that are to be determined, which represent soliton velocity, soliton frequency, wave number and phase constant, respectively. Next, $\psi(x,t)$ is a real function which represents the phase component of the soliton, while $\phi_j(\xi)$ are real functions that give the shape of the pulses. Substituting (3) and (4) into Eqs. (1) and (2), and after separating real and imaginary parts, the real parts are

$$\begin{aligned} & b_1\phi_2'''' + 3\left(a_1\kappa - 2b_1\kappa^2\right)\phi_2'' + \left(\beta_1 - a_1\kappa^3 + b_1\kappa^4\right)\phi_2 \\ & + (\alpha_1\kappa - \omega)\phi_1 + [c_1 - (\mu_1 + \gamma_1)\kappa]\phi_1^3 \\ & + (\sigma_1 + d_1)\phi_1\phi_2^2 + \xi_1\phi_1^5 + \eta_1\phi_1^3\phi_2^2 + \zeta_1\phi_1\phi_2^4 \\ & + l_1\phi_1^7 + m_1\phi_1^5\phi_2^2 + n_1\phi_1^3\phi_2^4 + p_1\phi_1\phi_2^6 = 0, \end{aligned} \tag{5}$$

and

$$\begin{aligned} & b_2\phi_1'''' + 3\left(a_2\kappa - 2b_2\kappa^2\right)\phi_1'' + \left(\beta_2 - a_2\kappa^3 + b_2\kappa^4\right)\phi_1 \\ & + (\alpha_2\kappa - \omega)\phi_2 + [c_2 - (\mu_2 + \gamma_2)\kappa]\phi_2^3 \\ & + (\sigma_2 + d_2)\phi_2\phi_1^2 + \xi_2\phi_2^5 + \eta_2\phi_2^3\phi_1^2 + \zeta_2\phi_2\phi_1^4 \\ & + l_2\phi_2^7 + m_2\phi_2^5\phi_1^2 + n_2\phi_2^3\phi_1^4 + p_2\phi_2\phi_1^6 = 0, \end{aligned} \tag{6}$$

while the imaginary parts are

$$\begin{aligned} & (a_1 - 4b_1\kappa)\phi_2''' + \left(4b_1\kappa^3 - 3a_1\kappa^2\right)\phi_2' \\ & + (\alpha_1 - v)\phi_1' - (3\gamma_1 + 2\theta_1 + \mu_1)\phi_1^2\phi_1' = 0, \end{aligned} \tag{7}$$

and

$$\begin{aligned} & (a_2 - 4b_2\kappa)\phi_1''' + \left(4b_2\kappa^3 - 3a_2\kappa^2\right)\phi_1' \\ & + (\alpha_2 - v)\phi_2' - (3\gamma_2 + 2\theta_2 + \mu_2)\phi_2^2\phi_2' = 0. \end{aligned} \tag{8}$$

Set

$$\phi_2(\xi) = \Pi\phi_1(\xi), \tag{9}$$

where $\Pi \neq 0$ or 1. Consequently, the real parts change to

$$\begin{gathered} b_1\Pi\phi_1'''' + 3\left(a_1\kappa - 2b_1\kappa^2\right)\Pi\phi_1'' \\ + \left[\left(\beta_1 - a_1\kappa^3 + b_1\kappa^4\right)\Pi + \alpha_1\kappa - \omega\right]\phi_1 \\ + \left[c_1 - (\mu_1 + \gamma_1)\kappa + (\sigma_1 + d_1)\Pi^2\right]\phi_1^3 \\ + \left(\xi_1 + \eta_1\Pi^2 + \zeta_1\Pi^4\right)\phi_1^5 \\ + \left(l_1 + m_1\Pi^2 + n_1\Pi^4 + p_1\Pi^6\right)\phi_1^7 = 0, \end{gathered} \tag{10}$$

and

$$\begin{gathered} b_2\phi_1'''' + 3\left(a_2\kappa - 2b_2\kappa^2\right)\phi_1'' \\ + \left[\beta_2 - a_2\kappa^3 + b_2\kappa^4 + (\alpha_2\kappa - \omega)\Pi\right]\phi_1 \\ + \left\{[c_2 - (\mu_2 + \gamma_2)\kappa]\Pi^3 + (\sigma_2 + d_2)\Pi\right\}\phi_1^3 \\ + \left(\xi_2\Pi^5 + \eta_2\Pi^3 + \zeta_2\Pi\right)\phi_1^5 \\ + \left(l_2\Pi^7 + m_2\Pi^5 + n_2\Pi^3 + p_2\Pi\right)\phi_1^7 = 0, \end{gathered} \tag{11}$$

while the imaginary parts become

$$\begin{gathered} (a_1 - 4b_1\kappa)\Pi\phi_1''' + \left[\left(4b_1\kappa^3 - 3a_1\kappa^2\right)\Pi + \alpha_1 - v\right]\phi_1' \\ - (3\gamma_1 + 2\theta_1 + \mu_1)\Pi_1^2\Pi_1' = 0, \end{gathered} \tag{12}$$

and

$$\begin{gathered} (a_2 - 4b_2\kappa)\phi_1''' + \left[\left(4b_2\kappa^3 - 3a_2\kappa^2\right) + (\alpha_2 - v)\Pi\right]\phi_1' \\ - (3\gamma_2 + 2\theta_2 + \mu_2)\Pi^3\phi_1^2\phi_1' = 0. \end{gathered} \tag{13}$$

The linearly independent principle applied to (12) and (13) gives the frequency of soliton as:

$$\begin{gathered} \kappa = \frac{a_j}{4b_j}, \quad a_j \neq 0, \quad b_j \neq 0 \\ \text{for} \quad j = 1, 2, \quad a_1b_2 = a_2b_1, \end{gathered} \tag{14}$$

the velocity of soliton as:

$$v = \left(4b_1\kappa^3 - 3a_1\kappa^2\right)\Pi + \alpha_1, \tag{15}$$

or

$$v = \frac{\left(4b_2\kappa^3 - 3a_2\kappa^2\right) + \Pi\alpha_2}{\Pi}, \tag{16}$$

and the constraint conditions:

$$3\gamma_j + 2\theta_j + \mu_j = 0 \quad \text{for} \quad j = 1, 2. \tag{17}$$

From (15) and (16), one obtains the following constraint condition:

$$\begin{gathered} 4\left(b_1\Pi^2 - b_2\right)\kappa^3 - 3\left(a_1\Pi^2 - a_2\right)\kappa^2 \\ + (\alpha_1 - \alpha_2)\Pi = 0. \end{gathered} \tag{18}$$

Equations (10) and (11) maintain the same form under the constraint conditions:

$$\begin{gathered} b_1\Pi = b_2, \\ (a_1 - 2b_1\kappa)\Pi = a_2 - 2b_2\kappa, \\ \left(\beta_1 - a_1\kappa^3 + b_1\kappa^4\right)\Pi + \alpha_1\kappa - \omega \\ = \beta_2 - a_2\kappa^3 + b_2\kappa^4 + (\alpha_2\kappa - \omega)\Pi, \\ c_1 - (\mu_1 + \gamma_1)\kappa + (\sigma_1 + d_1)\Pi^2 \\ = [c_2 - (\mu_2 + \gamma_2)\kappa]\Pi^3 + (\sigma_2 + d_2)\Pi, \\ \xi_1 + \eta_1\Pi^2 + \zeta_1\Pi^4 = \xi_2\Pi^5 + \eta_2\Pi^3 + \zeta_2\Pi, \\ l_1 + m_1\Pi^2 + n_1\Pi^4 + p_1\Pi^6 = l_2\Pi^7 + m_2\Pi^5 + n_2\Pi^3 + p_2\Pi. \end{gathered} \tag{19}$$

From (19), we have:

$$\begin{gathered} \omega = \frac{\beta_1\Pi - \beta_2 + (\alpha_1 - \alpha_2\Pi)\kappa - (a_1\Pi - a_2)\kappa^3}{1 - \Pi}, \\ \Pi = \frac{b_2}{b_1} = \frac{a_2 - 2b_2\kappa}{a_1 - 2b_1\kappa}, \quad b_1 \neq b_2 \quad \text{and} \quad a_1 \neq a_2. \end{gathered} \tag{20}$$

Now, we balance $\phi_1^{(iv)}$ and ϕ_1^7 in Eq. (10) to get the balance number $N = 2/3$. Since the balance number is not integer, we take into consideration the transformation

$$\phi_1(\xi) = [U(\xi)]^{\frac{2}{3}}, \tag{21}$$

where $U(\xi)$ is a new positive function of ξ. Substituting (21) into Eq. (10), we have a new equation

$$\begin{gathered} 144UU'^2U'' - 56U'^4 - 54U^2U''^2 - 72U^2U'U''' \\ + 54U^3U'''' + \frac{54(a_1\kappa - 2b_1\kappa^2)}{b_1}(3U^3U'' - U^2U'^2) \\ + \frac{81\left[\left(\beta_1 - a_1\kappa^3 + b_1\kappa^4\right)\Pi + \alpha_1\kappa - \omega\right]}{b_1\Pi}U^4 \\ + \frac{81\left(l_1 + m_1\Pi^2 + n_1\Pi^4 + p_1\Pi^6\right)}{b_1\Pi}U^8 \\ + \frac{81\left[c_1 - (\mu_1 + \gamma_1)\kappa + (\sigma_1 + d_1)\Pi^2\right]}{b_1\Pi}U^{\frac{16}{3}} \\ + \frac{81\left(\xi_1 + \eta_1\Pi^2 + \zeta_1\Pi^4\right)}{b_1\Pi}U^{\frac{20}{3}} = 0. \end{gathered} \tag{22}$$

For integrability, one must select

$$\begin{gathered} c_1 - (\mu_1 + \gamma_1)\kappa + (\sigma_1 + d_1)\Pi^2 = 0, \\ \xi_1 + \eta_1\Pi^2 + \zeta_1\Pi^4 = 0. \end{gathered} \tag{23}$$

Thus, from (19), one gets

$$\begin{gathered}[c_2-(\mu_2+\gamma_2)\kappa]\Pi^2+\sigma_2+d_2=0,\\ \xi_2\Pi^4+\eta_2\Pi^2+\zeta_2=0.\end{gathered} \tag{24}$$

Consequently, Eq. (22) changes to:

$$\begin{aligned}&54U^3U^{(iv)}+144UU'^2U''-56U'^4-54U^2U''^2\\ &-72U^2U'U'''+18\Delta_0\left(3U^3U''-U^2U'^2\right)\\ &+81\Delta_1U^4+81\Delta_7U^8=0,\end{aligned} \tag{25}$$

where

$$\left.\begin{aligned}\Delta_0&=\frac{3\left(a_1\kappa-2b_1\kappa^2\right)}{b_1},\\ \Delta_1&=\frac{\left(\beta_1-a_1\kappa^3+b_1\kappa^4\right)\Pi+\alpha_1\kappa-\omega}{b_1\Pi},\\ \Delta_7&=\frac{l_1+m_1\Pi^2+n_1\Pi^4+p_1\Pi^6}{b_1\Pi}.\end{aligned}\right\} \tag{26}$$

Now, Eq. (25) will be solved by using the methods as enumerated in the subsequent sections:

3. THE NEW AUXILIARY EQUATION METHOD

According to this method, Eq. (25) has the formal solution:

$$U(\xi)=\sum_{L=0}^{N}B_Lg^L(\xi), \tag{27}$$

where $g(\xi)$ satisfies:

$$g'^2(\xi)=\sum_{m=0}^{8}h_mg^m(\xi). \tag{28}$$

Here B_L and h_m are constants to be determined such that $B_N\neq 0$ and $h_8\neq 0$, where N is a positive integer. We determine the balancing number N of (27) by using the homogeneous balancing method as follows:

If $D(U)=N$, $D(U')=N+3$, $D(U'')=N+6$ and hence

$$D\left[U^sU^{(n)}\right]=N(s+1)+3n. \tag{29}$$

It is well known [19] that Eq. (28) has the following types of solutions:

Type 1: If $h_0=h_1=h_3=h_4=h_6=h_7=0$, $h_5^2-4h_2h_8>0$, $h_2>0$, then Eq. (28) gives the bright soliton solutions:

$$g(\xi)=\left(\frac{2\epsilon h_2}{\sqrt{h_5^2-4h_2h_8}\cosh\left(3\sqrt{h_2}\xi\right)-\epsilon h_5}\right)^{\frac{1}{3}}, \tag{30}$$

where $\epsilon=\pm1$.

Type 2: If $h_0=h_1=h_3=h_4=h_6=h_7=0$, $h_5^2-4h_2h_8<0$, $h_2>0$, then Eq. (28) admits the singular soliton solutions:

$$g(\xi)=\left(\frac{2\epsilon h_2}{\sqrt{-\left(h_5^2-4h_2h_8\right)}\sinh\left(3\sqrt{h_2}\xi\right)-\epsilon h_5}\right)^{\frac{1}{3}}, \tag{31}$$

where $\epsilon=\pm1$.

Type 3: If $h_0=h_1=h_3=h_4=h_6=h_7=0$, $h_5^2-4h_2h_8=0$, $h_2>0$, then Eq. (28) admits the dark soliton solutions as:

$$g(\xi)=\left\{-\frac{h_2}{h_5}\left[1\pm\tanh\left(\frac{3}{2}\sqrt{h_2}\xi\right)\right]\right\}^{\frac{1}{3}}, \tag{32}$$

and the singular soliton solutions as:

$$g(\xi)=\left\{-\frac{h_2}{h_5}\left[1\pm\coth\left(\frac{3}{2}\sqrt{h_2}\xi\right)\right]\right\}^{\frac{1}{3}}. \tag{33}$$

Balancing $U^3U^{(iv)}$ and U^8 in Eq. (25) by using (29), one recovers the balancing number $N=3$. From (27), one obtains the formal solution of Eq. (25) as:

$$U(\xi)=B_0+B_1f(\xi)+B_2f^2(\xi)+B_3f^3(\xi), \tag{34}$$

where B_l $(l=0,1,2,3)$ are constants to be determined such that $B_3\neq 0$. Substituting (34) and (28) with $h_0=h_1=h_3=h_4=h_6=h_7=0$ into Eq. (25), collecting all the coefficients of $[g(\xi)]^{m_1}[g(\xi)]^{m_2}$ $(m_1=0,1,\ldots,24,\ m_2=0,1)$ and setting these coefficients to zero, we have a set of algebraic equations which can be solved using the Maple to arrive at the following results:

$$\begin{gathered}B_0=B_1=B_2=0,\quad B_3=2\epsilon\left(-\frac{55h_8^2}{\Delta_7}\right)^{\frac{1}{4}},\\ h_2=-\frac{\Delta_0}{68},\quad h_5=0,\quad h_8=h_8,\end{gathered} \tag{35}$$

and

$$\Delta_1=\frac{16}{289}\Delta_0^2,\quad \Delta_7=\Delta_7,\quad \Delta_0=\Delta_0, \tag{36}$$

provided $\Delta_7<0$ and $\epsilon=\pm1$. Substituting (35) along with (30) and (31) into (34), one gets the following solutions of Eqs. (1) and (2):

(I) The bright soliton solutions:

$$q(x,t) = \left(-\frac{55\Delta_0^2}{289\Delta_7}\right)^{\frac{1}{6}} \times \operatorname{sech}^{\frac{2}{3}}\left[\frac{3}{2}\sqrt{-\frac{\Delta_0}{17}}(x - vt)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{37}$$

and

$$r(x,t) = \Pi\left(-\frac{55\Delta_0^2}{289\Delta_7}\right)^{\frac{1}{6}} \times \operatorname{sech}^{\frac{2}{3}}\left[\frac{3}{2}\sqrt{-\frac{\Delta_0}{17}}(x - vt)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{38}$$

(II) The singular soliton solutions:

$$q(x,t) = \left(-\frac{55\Delta_0^2}{289\Delta_7}\right)^{\frac{1}{6}} \times \operatorname{csch}^{\frac{2}{3}}\left[\frac{3}{2}\sqrt{-\frac{\Delta_0}{17}}(x - vt)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{39}$$

and

$$r(x,t) = \Pi\left(-\frac{55\Delta_0^2}{289\Delta_7}\right)^{\frac{1}{6}} \times \operatorname{csch}^{\frac{2}{3}}\left[\frac{3}{2}\sqrt{-\frac{\Delta_0}{17}}(x - vt)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{40}$$

provided $\Delta_7 < 0$ and $\Delta_0 < 0$. The solutions (37)–(40) exist along as the constraint condition (36) holds.

Furthermore, since $\operatorname{sech}(i\xi) = \sec(\xi)$ and $\operatorname{csch}(i\xi) = -i\csc(\xi)$; then from Eqs. (37)–(40), one achieves periodic wave solutions of Eqs. (1) and (2) as:

$$q(x,t) = \left(-\frac{55\Delta_0^2}{289\Delta_7}\right)^{\frac{1}{6}} \times \sec^{\frac{2}{3}}\left[\frac{3}{2}\sqrt{\frac{\Delta_0}{17}}(x - vt)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{41}$$

and

$$r(x,t) = \Pi\left(-\frac{55\Delta_0^2}{289\Delta_7}\right)^{\frac{1}{6}} \times \sec^{\frac{2}{3}}\left[\frac{3}{2}\sqrt{\frac{\Delta_0}{17}}(x - vt)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{42}$$

or

$$q(x,t) = \left(-\frac{55\Delta_0^2}{289\Delta_7}\right)^{\frac{1}{6}} \times \csc^{\frac{2}{3}}\left[\frac{3}{2}\sqrt{\frac{\Delta_0}{17}}(x - vt)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{43}$$

and

$$r(x,t) = \Pi\left(-\frac{55\Delta_0^2}{289\Delta_7}\right)^{\frac{1}{6}} \times \csc^{\frac{2}{3}}\left[\frac{3}{2}\sqrt{\frac{\Delta_0}{17}}(x - vt)\right]e^{i(-\kappa x+\omega t+\theta_0)}, \tag{44}$$

provided $\Delta_7 < 0$ and $\Delta_0 > 0$. The solutions (41)–(44) exist as long as the constraint condition (36) remains valid.

4. ADDENDUM TO KUDRYASHOV'S METHOD

For this method, Eq. (25) has the following solution structure:

$$U(\xi) = \sum_{L=0}^{N} \delta_L R^L(\xi), \tag{45}$$

where δ_L are parameters to be determined with $\delta_N \neq 0$, and $R(\xi)$ is the solution of:

$$R'^2(\xi) = R^2(\xi)\left[1 - \Gamma R^{2p}(\xi)\right]\ln^2 K, \quad 0 < K \neq 1, \tag{46}$$

here Γ is arbitrary constant, such that Eq. (46) has the solution:

$$R(\xi) = \left[\frac{4A}{4A^2 \exp_K(p\xi) + \Gamma \exp_K(-p\xi)}\right]^{\frac{1}{p}}, \tag{47}$$

where A is a nonzero constant, p is a positive integer and $\exp_K(p\xi) = K^{p\xi}$. It is easy to write (47) in the form of the combo bright-singular soliton solution as:

$$R(\xi) = \left[\frac{4A}{\left(4A^2 + \Gamma\right)\cosh(p\xi \ln K) + \left(4A^2 - \Gamma\right)\sinh(p\xi \ln K)}\right]^{\frac{1}{p}}. \tag{48}$$

Now, by balancing $U^3U^{(iv)}$ and U^8 in Eq. (25), one gets

$$N = p. \tag{49}$$

Next, we will discuss the following cases:

Case 1: If we choose $p = 1$, then $N = 1$. Thus, from (45) we conclude that Eq. (25) has the formal solution:

$$U(\xi) = \delta_0 + \delta_1 R(\xi), \tag{50}$$

where δ_0 and δ_1 are constants to be determined with $\delta_2 \neq 0$ while $R(\xi)$ satisfies the first order auxiliary ordinary differential equation (ODE):

$$R'^2(\xi) = R^2(\xi)\left[1 - \Gamma R^2(\xi)\right]\ln^2 K, \quad 0 < K \neq 1, \tag{51}$$

where Γ is a constant. Substituting (50) along with (51) into Eq. (25), collecting all the coefficients of each power of $[R(\xi)]^{m_1}\,[R'(\xi)]^{m_2}$ $(m_1 = 0,1,\ldots,8,\ m_2 = 0,1)$ and setting each of these coefficients to zero, we have a system of algebraic equations which can be solved by using the Maple to recover:

$$\delta_0 = 0, \quad \delta_2 = \epsilon\left(-\frac{880\Gamma^2}{81\Delta_7}\right)^{\frac{1}{4}}\ln K, \tag{52}$$

and

$$\Delta_0 = -\frac{68}{9}\ln^2 K, \quad \Delta_1 = \frac{256}{81}\ln^4 K, \tag{53}$$

provided $\Delta_7 < 0$ and $\epsilon = \pm 1$. Substituting (52) along with (48) into Eq. (50), one gets the combo bright-singular soliton solutions of Eqs. (1) and (2) in the forms:

$$q(x,t) = \left(-\frac{880\Gamma^2}{81\Delta_7}\right)^{\frac{1}{6}}\left[\frac{4A\ln K}{\left(4A^2+\Gamma\right)\cosh[(x-vt)\ln K] + \left(4A^2-\Gamma\right)\sinh[(x-vt)\ln K]}\right]^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{54}$$

and

$$r(x,t) = \Pi\left(-\frac{880\Gamma^2}{81\Delta_7}\right)^{\frac{1}{6}}\left[\frac{4A\ln K}{\left(4A^2+\Gamma\right)\cosh[(x-vt)\ln K] + \left(4A^2-\Gamma\right)\sinh[(x-vt)\ln K]}\right]^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}. \tag{55}$$

Especially, if we set $\Gamma = 4A^2$ in (54) and (55), then Eqs. (1) and (2) have the bright soliton solutions as:

$$q(x,t) = \left(-\frac{880}{81\Delta_7}\right)^{\frac{1}{6}} \times \{(\ln K)\,\mathrm{sech}[(x-vt)\ln K]\}^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{56}$$

and

$$r(x,t) = \Pi\left(-\frac{880}{81\Delta_7}\right)^{\frac{1}{6}} \times \{(\ln K)\,\mathrm{sech}[(x-vt)\ln K]\}^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{57}$$

while, if $\Gamma = -4A^2$, one gets the singular soliton solution of Eqs. (1) and (2) as:

$$q(x,t) = \left(-\frac{880}{81\Delta_7}\right)^{\frac{1}{6}} \times \{(\ln K)\,\mathrm{csch}[(x-vt)\ln K]\}^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}, \tag{58}$$

and

$$r(x,t) = \Pi\left(-\frac{880}{81\Delta_7}\right)^{\frac{1}{6}} \times \{(\ln K)\,\mathrm{csch}[(x-vt)\ln K]\}^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}. \tag{59}$$

The solutions (54)–(59) exist under the conditions (53).

Case 2: If we choose $p = 2$, then $N = 4$. Thus, we derive from (41) that Eq. (21) has the formal solution:

$$U(\xi) = \delta_0 + \delta_1 R(\xi) + \delta_2 R^2(\xi), \tag{60}$$

where δ_0, δ_1 and δ_2 are constants to be determined, such that $\delta_2 \neq 0$ and the function $R(\xi)$ satisfies the first order auxiliary ODE

$$R'^2(\xi) = R^2(\xi)\left[1 - \Gamma R^4(\xi)\right]\ln^2 K, \quad 0 < K \neq 1, \tag{61}$$

where Γ is a constant. Substituting (60) and (61) into Eq. (25), collecting all the coefficients of each power of $[R(\xi)]^{m_1}\,[R'(\xi)]^{m_2}$ $(m_1 = 0,1,2,\ldots,16,\ m_2 = 0,1)$ and setting each of these coefficients to zero, we have a system of algebraic equations which can be solved by using the Maple to obtain:

$$\delta_0 = \delta_1 = 0, \quad \delta_2 = \epsilon\left(-\frac{14080\Gamma^2}{81\Delta_7}\right)^{\frac{1}{4}}\ln K, \tag{62}$$

and

$$\Delta_0 = -\frac{272}{9}\ln^2 K, \quad \Delta_1 = \frac{4096}{81}\ln^4 K, \tag{63}$$

provided $\Delta_7 < 0$ and $\epsilon = \pm 1$. Substituting (62) along with (48) into Eq. (60), we have the combo bright-singular soliton solutions of Eqs. (1) and (2) in the forms:

$$q(x,t) = \left(-\frac{14\,080\Gamma^2}{81\Delta_7}\right)^{\frac{1}{6}} \left[\frac{4A\ln K}{\left(4A^2+\Gamma\right)\cosh\left[2(x-vt)\ln K\right]+\left(4A^2-\Gamma\right)\sinh\left[2(x-vt)\ln K\right]}\right]^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}, \quad (64)$$

and

$$r(x,t) = \Pi\left(-\frac{14\,080\Gamma^2}{81\Delta_7}\right)^{\frac{1}{6}} \left[\frac{4A\ln K}{\left(4A^2+\Gamma\right)\cosh\left[2(x-vt)\ln K\right]+\left(4A^2-\Gamma\right)\sinh\left[2(x-vt)\ln K\right]}\right]^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}. \quad (65)$$

In particular, if we set $\Gamma = 4A^2$ in (64) and (65), then one gets the bright soliton solution of Eqs. (1) and (2) as:

$$q(x,t) = \left(-\frac{14\,080\Gamma^2}{81\Delta_7}\right)^{\frac{1}{6}} \times \left\{(\ln K)\,\mathrm{sech}[2(x-vt)\ln K]\right\}^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}, \quad (66)$$

and

$$r(x,t) = \Pi\left(-\frac{14\,080\Gamma^2}{81\Delta_7}\right)^{\frac{1}{6}} \times \left\{(\ln K)\,\mathrm{sech}[2(x-vt)\ln K]\right\}^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}, \quad (67)$$

while, if $\Gamma = -4A^2$, we have the singular soliton solution of Eqs. (1) and (2) as:

$$q(x,t) = \left(-\frac{14\,080\Gamma^2}{81\Delta_7}\right)^{\frac{1}{6}} \times \left\{(\ln K)\,\mathrm{csch}[2(x-vt)\ln K]\right\}^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}, \quad (68)$$

and

$$r(x,t) = \Pi\left(-\frac{14\,080\Gamma^2}{81\Delta_7}\right)^{\frac{1}{6}} \times \left\{(\ln K)\,\mathrm{csch}[2(x-vt)\ln K]\right\}^{\frac{2}{3}} e^{i(-\kappa x+\omega t+\theta_0)}. \quad (69)$$

The solutions (64)–(69) exist under the conditions (63).

Similarly, we can find many other solutions by choosing other values for p and N.

5. CONCLUSIONS

The current paper retrieved bright dark and singular soliton solutions in fiber Bragg gratings that was studied with polynomial law of nonlinear refractive index having cubic-quartic form of dispersive reflectivity. The results are impressive and indeed promising for future work. The model is yet to be studied for location of the conservation laws [11–20]. Additional solutions can be retrieved by the aid of Lie symmetry and other such powerful integration schemes. The model is yet to be studied with time-dependent coefficients as well as with fractional temporal evolution. The results of those research activities will be reported with time.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. J. Atai and B. Malomed, Phys. Lett. A **284**, 247 (2001).
2. J. Atai and B. Malomed, Phys. Lett. A **342**, 404 (2005).
3. A. Biswas, J. Vega-Guzman, M. F. Mahmood, S. Khan, Q. Zhou, S. P. Moshokoa, and M. Belic, Optik **182**, 119 (2019).
4. A. Biswas, M. Ekici, A. Sonmezoglu, and M. R. Belic, Optik **182**, 88 (2019).
5. A. Biswas, M. Ekici, A. Sonmezoglu, and M. R. Belic, Optik **183**, 595 (2019).
6. S. A. M. S. Chowdhury and J. Atai, IEEE J. Quantum Electron. **50**, 458 (2014).
7. S. A. M. S. Chowdhury and J. Atai, J. Mod. Opt. **63**, 2238 (2016).
8. S. A. M. S. Chowdhury and J. Atai, Sci. Rep. **7**, 4021 (2017).
9. S. Dasanayaka and J. Atai, Phys. Lett. A **375**, 225 (2010).
10. M. J. Islam and J. Atai, Nonlin. Dyn. **87**, 1693 (2017).
11. Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic, USA, 2003).
12. N. A. Kudryashov, Chin. J. Phys. **66**, 401 (2020).
13. D. R. Neill, J. Atai, and B. A. Malomed, J. Opt. A **10**, 085105 (2008).
14. X. T. Tran and N. N. Rosanov, Opt. Spectrosc. **104**, 558 (2008).
15. X. T. Tran and N. N. Rosanov, Opt. Spectrosc. **105**, 393 (2008).
16. Y. Yildirim, A. Biswas, A. H. Kara, M. Ekici, S. Khan, and M. R. Belic, Semicond. Phys. Quantum Electron. Optoelectron. **24**, 64 (2021).
17. E. M. E. Zayed, M. El-Horbaty, and M. E. M. Alngar, Optik **220**, 165121 (2020).
18. E. M. E. Zayed, M. E. M. Alngar, A. Biswas, M. Ekici, H. Triki, A. K. Alzahrani, and M. R. Belic, Optik **204**, 164096 (2020).
19. E. M. E. Zayed, M. E. M. Alngar, M. M. El-Horbaty, A. Biswas, M. Ekici, Q. Zhou, and M. R. Belic, Indian J. Phys. **95**, 109 (2021).
20. E. M. E. Zayed, R. M. A. Shohib, M. E. M. Alngar, A. Biswas, M. Ekici, S. Khan, A. K. Alzahrani, and M. R. Belic, Ukr. J. Phys. Opt. **22**, 38 (2021).