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Dynamical system of optical soliton parameters for anti-cubic and generalized anti-cubic nonlinearities with super-Gaussian and super-sech pulses

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The parameter dynamics of solitons, propagating through optical fibers, is emerged from the usage of variational principle. The anti-cubic nonlinearity and its generalized version are studied. This study reveals that the center position does not affect the dynamics of different parameters and only soliton power and linear momentum are conserved quantities.

Keywords: super-sech pulse, super-Gaussian, anti-cubic, variational approach.

1. Introduction

The nonlinear Schrödinger equation (NLSE) appears in a number of areas in mathematical physics and they range from quantum optics, fluid dynamics, plasma physics, nuclear physics, biochemistry and various others [1]. The challenge is to find an exact solution for these partial differential equations containing nonlinearities by means of an efficient method. A wide range of integration algorithms locate exact solutions to NLSE as indicated in the literature. Among them are such as Hirota's bilinear method, Kudryashov's scheme, the G'/G-expansion, auxiliary equation approach, the homotopy perturbation theory, the Lie group analysis, the Jacobi's elliptic function expansion [2–5] and these are just a few droplets in the ocean. Additionally, the variational method is one of the powerful methods for securing the soliton parameter dynamics with or without perturbation terms. Other methods to obtain these parameter dynamical systems are the moment method, collective variables and soliton perturbation theory. The aim of this paper is to apply Anderson's method to obtain soliton parameter evolution with generalized anti-cubic (GAC) nonlinearity and anti-cubic (AC) nonlinearity [6, 7].

The soliton pulse types that will be incorporated in this work are super-Gaussian (SG) and super-sech (SS) types. The Gaussian and sech pulses are fundamental soliton solutions of the governing model that maintains logarithmic law of nonlinearity or others. These models have produced a better description of coherent radiation like laser beams. Added nonlinear terms to the standard form of Schrödinger's equation require taking into account several parameters in the trial function. AC nonlinearity first appeared during 2003 and this nonlinear form gained popularity and several journals were flooded with a wide variety of results that obeyed such a nonlinear form [8, 9]. This AC nonlinearity is a generalized version of the Kerr nonlinear form [10–13]. In [14] the authors obtained the integration of the resonant NLSE along with perturbation terms and the anti-cubic nonlinear form. Three methods were implemented to extract the analytical soliton solutions. The study of the effect of nonlinearities on pulse dynamics has been the subject of several publications in recent years. These are the parabolic law, Kerr law and the logarithmic law. This Kerr law accounts for the cubic nonlinear form. Most optical fibers that are commercially available maintain this Kerr nonlinear form.

The outline of the present work is as follows: in Sections 2 and 3, we use the variational formulation for constructing the trial Lagrangian. Using SG and the SS trial functions in Section 4, we obtain the optical soliton parameter evolution. The conclusive statements are written up in Section 5.

2. NLSE with AC nonlinearity

The NLSE is written as [8–17]:

$$iq_z + aq_{tt} + \left(\frac{b_1}{|q|^4} + b_2|q|^2 + b_3|q|^4\right)q = i\alpha q_t + i\lambda(|q|^2q)_t + i\theta(|q|^2)_t q \qquad (1)$$

In (1), the complex-valued function $q(z, t)$ accounts for the soliton profile. The independent variables t and z are respectively the temporal and spatial coordinates. The first term accounts for linear spatial evolution of the soliton pulses where $i = \sqrt{-1}$. The coefficient a is from chromatic dispersion. The last three terms are collectively referred to as AC nonlinearity. On the right side of (1), the perturbation terms are included. The coefficient of α is from inter-modal dispersion and the coefficients λ and θ are due to self -steepening effect and nonlinear dispersion effect, respectively. If $b_1 = 0$, one recovers NLSE in parabolic law nonlinear fibers.

2.1. Mathematical formulation

The Lagrangian density L of this system without the perturbation terms is of the form [6, 7, 18–20]:

$$L = \frac{i}{4}(q_z q^* - q_z^* q) - \frac{a}{2}|q_t|^2 - \frac{b_1}{2|q|^2} + \frac{b_2}{4}|q|^4 + \frac{b_3}{6}|q|^6 \tag{2}$$

Therefore, the average Lagrangian L_g is defined as

$$L_g = \int_{-\infty}^{+\infty} L \, dt \tag{3}$$

and the conserved quantities are

$$E = \int_{-\infty}^{+\infty} |q|^2 dt \tag{4}$$

$$M = ia\int_{-\infty}^{+\infty} (q_t q^* - q_t^* q) dt \tag{5}$$

$$H = \int_{-\infty}^{+\infty} \left(a|q_t|^2 + \frac{b_1}{|q|^2} + \frac{b_3}{3}|q|^6 \right) dt \tag{6}$$

where M represents linear momentum while E refers to soliton power and H signifies Hamiltonian [16, 17]. The solution of this system is

$$q(X_J, t) = X_1 f\left(\frac{t - X_2}{X_3}\right) \exp\left\{ i\left[\frac{x_4}{2}(t - X_2)^2 + X_5(t - X_2) + X_6\right]\right\}, \quad J = \{1, 2...6\} \tag{7}$$

where f signifies the shape of the pulse that is going to be the SS, $f(\tau)$= $\mathrm{sech}^m(\tau)$ or the SG, $\exp(-\tau^{2m})$. X_1, X_2, X_3, X_4, X_5, X_6 are respectively the soliton amplitude, the center of the pulse, the width of the pulse, the chirp, the frequency, the phase of the pulse and are the function of z. Next, m is the parameter of the SS or SG pulse.

In the Lagrangian variational algorithm, the integral is defined as:

$$\varphi_{i,j,k} = \int_{-\infty}^{+\infty} \tau^i f^j(\tau)\left(\frac{\mathrm{d}f}{\mathrm{d}\tau}\right)^k \mathrm{d}\tau \tag{8}$$

The Lagrangian (2) along with (7) becomes

$$\begin{aligned} L_g = & -\frac{q}{4}X_1^2 X_4' X_3^3 + \frac{p}{2}\left(X_1^2 X_5 X_2' X_3 - \frac{p}{2}X_1^2 X_6'\right)X_3 + \frac{b_2 d}{4}X_1^4 X_3 \\ & - \frac{b_1 b}{2}\frac{X_3}{X_1^2} - \frac{1}{2}\frac{X_1^2 a(r + pX_3^2 X_5^2 + qX_3^4 X_4^2)}{X_3} + \frac{b_3 h}{6}X_1^6 X_3 \end{aligned} \tag{9}$$

Here, $p = \varphi_{0,2,0}$, $q = \varphi_{2,2,0}$, $r = \varphi_{0,0,2}$, $b = \varphi_{0,-2,0}$, $d = \varphi_{0,4,0}$, and $h = = \varphi_{0,6,0}$.

The integrals of motion are:

$$E = pX_1^2 X_3 \tag{10}$$

$$M = -2apX_1^2 X_3 X_5 \tag{11}$$

while the Hamiltonian is given by:

$$H = \frac{X_1^2 a(r + pX_3^2 X_5^2 + qX_3^4 X_4^2)}{X_3} + bb_1\frac{X_3}{X_1^2} - \frac{b_3 h}{3}X_1^6 X_3 \tag{12}$$

We now consider the perturbed system whose equation is given by:

$$iq_z + aq_{tt} + \left(\frac{b_1}{|q|^4} + b_2|q|^2 + b_3|q|^4\right)q = i\varepsilon R\left[f, f^*\right] \tag{13}$$

where R stands for the perturbation terms while ε implies to the perturbation parameter [4]. The Euler–Lagrange equation [19, 20] is modified to

$$\frac{\partial L_g}{\partial X_j(z)} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial L_g}{\partial \dot{X}_j(z)} = i\varepsilon\int_{-\infty}^{+\infty}\left(R\frac{\partial q^*}{\partial X_j} - R^*\frac{\partial q}{\partial X_j}\right)\mathrm{d}t \tag{14}$$

By setting,

$$\psi_j = i\varepsilon\int_{-\infty}^{+\infty}\left(R\frac{\partial f^*}{\partial X_j} - R^*\frac{\partial f}{\partial X_j}\right) \tag{15}$$

and substituting the equation (13) with f in ψ_j leads to

$$\dot{X}_1 = -\frac{aqX_1^2X_3^3X_4 + \psi_4}{X_1X_3^3q} + \frac{3\psi_6}{2pX_1X_3} \tag{16a}$$

$$\dot{X}_2 = \frac{2(apX_1^2X_3X_5 + \psi_5)}{pX_1^2X_3} \tag{16b}$$

$$\dot{X}_3 = \frac{(2apX_1^2X_3^3X_4 + \psi_4)}{qX_1^2X_3^2} - \frac{\psi_6}{pX_1^2} \tag{16c}$$

$$\begin{aligned}\dot{X}_4 = {} & \frac{aqX_1^4X_3^4X_4^2 - hX_1^8X_3^2 + apX_1^4X_3^2X_5^2 - dX_1^6X_3^2 + arX_1^4 - \psi_1X_1^3X_3 - bX_3^2}{qX_1^4X_3^4} \\ & - \frac{1}{qX_1^2X_3^2}\left\{2\left[aX_1^2(2qX_3^2X_4^2 + pX_5^2) - \frac{aX_1^2(qX_3^4X_4^2 + pX_3^2X_5^2 + r)}{2X_3^2}\right]\right\} \\ & - \frac{1}{qX_1^2X_3^2}\left[2\left(\frac{b}{2X_1^2} - \frac{d}{4}X_1^4 - \frac{h}{6}X_1^6 - \psi_3\right)\right]\end{aligned} \tag{16d}$$

$$\dot{X}_5 = -\frac{2\psi_2}{pX_1X_3} - \frac{X_5(aqX_1^2X_3^3X_4 + \psi_4)}{qX_1^2X_3^3} - \frac{X_5\psi_6}{pX_1^2X_3} \tag{16e}$$

$$\begin{aligned}\dot{X}_6 = {} & \frac{3(aqX_1^4X_3^4X_4^2 - hX_1^8X_3^2 + apX_1^4X_3^2X_5^2 - dX_1^6X_3^2 + arX_1^4 - \psi_1X_1^3X_3 - bX_3^2)}{2p\,X_1^4X_3^2} \\ & + \frac{1}{pX_1^2}\left[aX_1^2(2qX_3^2X_4^2 + pX_5^2) - \frac{aX_1^2(qX_3^4X_4^2 + pX_3^2X_5^2 + r)}{2X_3^2}\right] \\ & + \frac{1}{pX_1^2}\left(\frac{b}{2X_1^2} - \frac{d}{4}X_1^4 - \frac{h}{6}X_1^6 - \psi_3\right)\end{aligned} \tag{16f}$$

This soliton parameter dynamics shows that the center position do not affect the dynamics of different parameters. Linear momentum and soliton power are obtained for AC nonlinearity.

3. NLSE with GAC nonlinearity

The NLSE is written as

$$iq_z + aq_{tt} + \left(\frac{b_1}{|q|^{2n+2}} + b_2|q|^{2n} + b_3|q|^{2n+2}\right)q = i\alpha q_t + i\lambda(|q|^2 q)_t + i\theta(|q|^2)_t q \tag{17}$$

In (17), the notations and their physical correspondence are the same as in for Eq. (1). The GAC parameter is n which makes (17) condense to (1) for $n = 1$. The remaining parameters interpret the same as in (1). Likewise, the case when $b_1 = 0$, NLSE with dual-power law nonlinearity falls out.

3.1. Mathematical formulation

The Lagrangian density L_0 of this system without the perturbation terms is of the form

$$L_0 = \frac{i}{4}(q_z q^* - q_z^* q) - \frac{a}{2}|q_t|^2 - \frac{b_1}{2n|q|^{2n}} + \frac{b_2}{2n+2}|q|^{2n+2} + \frac{b_3}{2n+4}|q|^{2n+4} \tag{18}$$

Therefore, the average Lagrangian L_g is defined as

$$L_g = \int_{-\infty}^{+\infty} L_0 \mathrm{d}t \tag{19}$$

and the conserved quantities are

$$E = \int_{-\infty}^{+\infty} |q|^2 \mathrm{d}t \tag{20}$$

$$M = ia\int_{-\infty}^{+\infty} (q_t q^* - q_t^* q)\mathrm{d}t \tag{21}$$

$$H = \int_{-\infty}^{+\infty}\left(a|q_t|^2 + \frac{b_1}{n|q|^{2n}} - \frac{b_3}{n+2}|q|^{2n+4}\right)\mathrm{d}t \tag{22}$$

where M represents linear momentum while E refers to soliton power and H signifies Hamiltonian [16, 17]. The solution of this system is

$$q(X_J, t) = X_1 f\left(\frac{t - X_2}{X_3}\right)\exp\left\{i\left[\frac{x_4}{2}(t - X_2)^2 + X_5(t - X_2) + X_6\right]\right\}, \quad J = \{1, 2...6\} \tag{23}$$

In the Lagrangian variational algorithm, the integral is defined:

$$\varphi_{i,j,k} = \int_{-\infty}^{+\infty} \tau^i f^j(\tau)\left(\frac{\mathrm{d}f}{\mathrm{d}\tau}\right)^k \mathrm{d}\tau \tag{24}$$

The Lagrangian (10) along with (7) becomes

$$L_{\mathrm{g}} = -\frac{q}{4}X_1^2 X_4' X_3^3 + \frac{p}{2}\left(X_1^2 X_5 X_2' X_3 - \frac{p}{2}X_1^2 X_6'\right)X_3 + \frac{b_2 d}{2n+2}X_1^{2n+2}X_3 + \frac{b_3 h}{2n+4}X_1^{2n+4}X_3 - \frac{b_1 b}{2n}\frac{X_3}{X_1^{2n}} - \frac{1}{2}\frac{X_1^2 a(r + pX_3^2 X_5^2 + qX_3^4 X_4^2)}{X_3} \tag{25}$$

with, $p = \varphi_{0,2,0}$, $q = \varphi_{2,2,0}$, $r = \varphi_{0,0,2}$, $b = \varphi_{0,-2n,0}$, $d = \varphi_{0,2n+2,0}$, and $h = \varphi_{0,2n+4,0}$.

The integrals of motion are:

$$E = pX_1^2 X_3 \tag{26}$$

$$M = -2apX_1^2 X_3 X_5 \tag{27}$$

while the Hamiltonian is given by

$$H = \frac{X_1^2 a(r + pX_3^2 X_5^2 + qX_3^4 X_4^2)}{X_3} + \frac{bb_1}{n}\frac{X_3}{X_1^{2n}} - \frac{b_3 h}{n+2}X_1^{2n+4}X_3 \tag{28}$$

We now consider the perturbed system whose equation is given by [19, 20]

$$iq_z + aq_{tt} + \left(\frac{b_1}{|q|^{2n+2}} + b_2|q|^{2n} + b_3|q|^{2n+2}\right)q = i\varepsilon R\left[f, f^*\right] \tag{29}$$

The Euler–Lagrange's equation is modified to

$$\frac{\partial L_{\mathrm{g}}}{\partial X_j(z)} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial L_{\mathrm{g}}}{\partial \dot{X}_j(z)} = i\varepsilon\int_{-\infty}^{+\infty}\left(R\frac{\partial q^*}{\partial X_j} - R^*\frac{\partial q}{\partial X_j}\right)\mathrm{d}t \tag{30}$$

By setting,

$$\psi_j = i\varepsilon\int_{-\infty}^{+\infty}\left(R\frac{\partial f^*}{\partial X_j} - R^*\frac{\partial f}{\partial X_j}\right)\mathrm{d}t \tag{31}$$

and substituting the Eq. (13) with f in ψ_j leads to

$$\dot{X}_1 = -\frac{aqX_1^2X_3^3X_4+\psi_4}{X_1X_3^3q}+\frac{3\psi_6}{2pX_1X_3} \tag{32a}$$

$$\dot{X}_2 = \frac{2(apX_1^2X_3X_5+\psi_5)}{pX_1^2X_3} \tag{32b}$$

$$\dot{X}_3 = \frac{(2apX_1^2X_3^3X_4+\psi_4)}{qX_1^2X_3^2}-\frac{\psi_6}{pX_1^2} \tag{32c}$$

$$\begin{aligned}\dot{X}_4 = {} & \frac{1}{qX_1X_3^3}\left[\frac{aX_1^2(qX_3^4X_4^2+pX_3^2X_5^2+r)}{X_3}-\frac{bX_3}{X_1^{2n+1}}\right.\\ & \left.-dX_1^{2n+1}X_3-hX_1^{2n+3}X_3-\psi_1\right]\\ & -\frac{1}{qX_1^2X_3^2}\left\{2\left[aX_1^2(2qX_3^2X_4^2+pX_5^2)-\frac{aX_1^2(qX_3^4X_4^2+pX_3^2X_5^2+r)}{2X_3^2}\right]\right\}\\ & -\frac{1}{qX_1^2X_3^2}\left[2\left(\frac{b}{2nX_1^{2n}}-\frac{d}{2n+2}X_1^{2n+2}-\frac{h}{2n+4}X_1^{2n+4}-\psi_3\right)\right]\end{aligned} \tag{32d}$$

$$\dot{X}_5 = -\frac{2\psi_2}{pX_1X_3}-\frac{X_5(aqX_1^2X_3^3X_4+\psi_4)}{qX_1^2X_3^3}-\frac{X_5\psi_6}{pX_1^2X_3} \tag{32e}$$

$$\begin{aligned}\dot{X}_6 = {} & -\frac{3}{2pX_1X_3}\left[\frac{aX_1^2(qX_3^4X_4^2+pX_3^2X_5^2+r)}{X_3}-\frac{bX_3}{X_1^{2n+1}}\right.\\ & \left.-dX_1^{2n+1}X_3-hX_1^{2n+3}X_3-\psi_1\right]\\ & +\frac{1}{pX_1^2}\left[aX_1^2(2qX_3^2X_4^2+pX_5^2)-\frac{aX_1^2(qX_3^4X_4^2+pX_3^2X_5^2+r)}{2X_3^2}\right]+\end{aligned}$$

$$+\frac{1}{pX_1^2}\left(\frac{b}{2nX_1^{2n}}-\frac{d}{2n+2}X_1^{2n+2}-\frac{h}{2n+4}X_1^{2n+4}-\psi_3\right)$$
$$+\frac{2X_5(apX_1^2X_3X_5+\psi_5)}{pX_1^2X_3} \tag{32f}$$

This soliton parameter dynamics shows that the center position do not affect the dynamics of different parameters. Linear momentum and soliton power are obtained for GAC nonlinearity.

4. SS and SG pulse dynamics

For a pulse of SG type and SS given respectively by: $f(\tau) = \exp(-\tau^{2m})$ and $f(\tau) = \operatorname{sech}^m(\tau)$ (here, the parameter $m > 0$), different integrals are given in Table 1, and

$${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; x) = \sum_{k=0}^{n}\frac{\prod_{j=1}^{p}(a_j)_k x^k}{\prod_{j=1}^{q}(b_j)_k k!} \tag{33}$$

with

$$(p)_n = \begin{cases} 1, & n = 0 \\ p(p+1)...(p+n-1), & n > 0 \end{cases} \tag{34}$$

while $B(l, m)$ and $\Gamma(x)$ are beta and gamma functions, respectively. The domain restriction for gamma function prompts the constraint $m > 1/4$.

Next, ψ_j terms are given for $m = 2$ in Table 2.

5. Conclusions

The variational method is applied to study NLSE with AC and GAC nonlinearities. This method transforms NLSE into a dynamical system of soliton parameters. The dynamics of the soliton center, amplitude, chirp, width, phase and frequency is fully revealed. This study reveals that the center position does not affect the dynamics of the remaining parameters. Linear momentum and soliton power conserved quantities are recovered for AC and GAC nonlinearities while Hamiltonian conserved quantity does not exist for these nonlinear forms since the corresponding improper integral is rendered to be divergent. In future, the plan is to recover soliton parameter dynamics using variational principle for several models that have recently been proposed [21–30]. The results of such research activities will be reported all across the board.

T a b l e 1. Values of different integrals $\varphi_{i,j,k}$ for SG and SS pulses from (8).

$\varphi_{i,j,k}$	SG	SS
$\varphi_{2,2,0}$	$\frac{2^{(2m-3)/2m}}{2m}\Gamma\left(\frac{3}{2m}\right)$	$\frac{2^{2m-1}}{m^3}\,{}_4F_3(m, m, m, 2m; 1+m, 1+m; -1)$
$\varphi_{0,0,2}$	$\frac{2m}{2^{(2m-1)/2m}}\Gamma\left(\frac{4m-1}{2m}\right)$	$-m^2\left[\frac{2mB\left(m, \frac{1}{2}\right)}{2m+1} - \frac{2^{2m-1}(m+1)\Gamma^2(m)}{(2m+1)\Gamma(2m)} - \frac{2^{2+m}+1}{2+m}\,{}_2F_1(2+m, 2+2m; 3+m; -1)\right]$
$\varphi_{0,2,0}$	$\frac{1}{2^{1/2m}m}\Gamma\left(\frac{1}{2m}\right)$	$B\left(m, \frac{1}{2}\right)$
$\varphi_{0,4,0}$	$\frac{4}{2^{(2m+1)/m}m}\Gamma\left(\frac{1}{m}\right)$	$B\left(2m, \frac{1}{2}\right)$
$\varphi_{0,6,0}$	$\frac{1}{6^{1/2m}m}\Gamma\left(\frac{1}{2m}\right)$	$B\left(3m, \frac{1}{2}\right)$
$\varphi_{0,-2,0}$	Divergent integral	Divergent integral
$\varphi_{0,-2n,0}$	Divergent integral	Divergent integral
$\varphi_{0,2n,0}$	$\frac{1}{(2n)^{1/2m}m}\Gamma\left(\frac{1}{2m}\right)$	$B\left(nm, \frac{1}{2}\right)$
$\varphi_{0,2n+2,0}$	$\frac{1}{(2n+2)^{1/2m}m}\Gamma\left(\frac{1}{2m}\right)$	$B\left((n+1)m, \frac{1}{2}\right)$
$\varphi_{0,2n+4,0}$	$\frac{1}{(2n+4)^{1/2m}m}\Gamma\left(\frac{1}{2m}\right)$	$B\left((n+2)m, \frac{1}{2}\right)$

Table 2. Values of different perturbations terms ψ_j for SG and SS pulses from (31).

ψ_j	SG	SS
ψ_1	$-\frac{1}{2}\sqrt{2\pi}\,\alpha X_1 X_3 X_5 - \frac{1}{2}\sqrt{\pi}\,\lambda X_1^3 X_5$	$-\frac{8}{3}\alpha X_1 X_3 X_5 - \frac{64}{35}\lambda X_1^3 X_5$
ψ_2	$\frac{1}{4}\sqrt{\pi}\,\lambda X_1^4 X_3 X_4 + \frac{1}{4}\sqrt{\pi}\,\theta X_1^4 X_3 X_4$	$\frac{32}{35}\lambda X_1^4 X_3 X_4 + \frac{32}{35}\theta X_1^4 X_3 X_4$
ψ_3	$-\frac{1}{4}\sqrt{2\pi}\,\alpha X_1^2 X_5 - \frac{1}{8}\sqrt{\pi}\,\lambda X_1^4 X_5$	$-\frac{4}{3}\alpha X_1^2 X_5 - \frac{16}{35}\lambda X_1^4 X_5$
ψ_4	0	0
ψ_5	$-\frac{1}{2}\sqrt{2\pi}\,\alpha X_1^2 X_3 - \frac{3}{4}\sqrt{\pi}\,\lambda X_1^4 X_3 - \frac{1}{2}\sqrt{\pi}\,\theta X_1^4 X_3$	$-\frac{4}{3}\alpha X_1^2 X_3 - \frac{48}{35}\lambda X_1^4 X_3 - \frac{32}{35}\theta X_1^4 X_3$
ψ_6	0	0

References

[1] Agrawal G.P., *Nonlinear Fiber Optics*, Academic Press, San Diego, CA, USA, 2003.

[2] Biswas A., Sonmezoglu A., Ekici M., Alshomrani A.S., Belic M., *Optical solitons with Kudryashov's equation by F-expansion*, Optik **199**, 2019, article 163338, DOI: 10.1016/j.ijleo.2019.163338.

[3] Biswas A., Ekici M., Sonmezoglu A., Alshomrani A., Belic M., *Optical solitons with Kudryashov's equation by extended trial function*, Optik **202**, 2020, article 163290, DOI: 10.1016/j.ijleo.2019.163290.

[4] Ayela A., Edah G., Elloh C., Djossou G., *Super-sech soliton dynamics in optical metamaterials with generally parabolic law of nonlinearity using Lagrangian variational method*, Physical Science International Journal **21**(3), 2019, pp. 1–9, DOI: 10.9734/psij/2019/v21i330109.

[5] Ayela A.M., Edah G., Biswas A., Ekici M., Alzahrani A.K., Belic M.R., *Chirped super-Gaussian and super-sech pulse parameter dynamics with DWDM topology by variational principle*, Optik **206**, 2020, article 164344, DOI: 10.1016/j.ijleo.2020.164344.

[6] Shwetanshumala S., *Bistable bright optical spatial solitons due to charge drift and diffusion of various orders in photovoltaic photorefractive media under closed-circuit conditions*, Zeitschrift für Naturforschung A **71**(2), 2016, pp. 175–184, DOI: 10.1515/zna-2015-0409.

[7] Anderson D., Lisak M., Berntson A., *A variational approach to nonlinear evolution equations in optics*, Pramana **57**(5–6), 2001, pp. 917–936, DOI: 10.1007/s12043-001-0006-z.

[8] Biswas A., Ekici M., Sonmezoglu A., Zhou Q., Alshomrani A.S., Moshokoa S.P., Belic M., *Solitons in optical metamaterials with anti-cubic nonlinearity*, The European Physical Journal Plus **133**(5), 2018, article 204, DOI: 10.1140/epjp/i2018-12046-6.

[9] Biswas A., Konar S., *Introduction to non-Kerr Law Optical Solitons*, CRC Press, Boca Raton, FL, USA, 2006.

[10] Ekici M., Sonmezoglu A., Zhou Q., Moshokoa S.P., Ullah M.Z., Arnous A.H., Biswas A., Belic M., *Analysis of optical solitons in nonlinear negative-indexed materials with anti-cubic nonlinearity*, Optical and Quantum Electronics **50**(2), 2018, article 75, DOI: 10.1007/s11082-018-1341-3.

[11] Fujioka J., Cortes E., Perez-Pascual R., Rodríguez R.F., Espinosa A., Malomed B.A., *Chaotic solitons in the quadratic-cubic nonlinear Schrödinger equation under nonlinearity management*, Chaos **21**(3), 2011, article 033120, DOI: 10.1063/1.3629985.

[12] Saha M., Sarma A.K., *Modulation instability in nonlinear metamaterials induced by cubic–quintic nonlinearities and higher order dispersive effects*, Optics Communications **291**, 2013, pp. 321–325, DOI: 10.1016/j.optcom.2012.11.011.

[13] Zhou Q., Zhu Q., Liu Y., Biswas A., Bhrawy A.H., Khan K.R., Mahmood M.F., Belic M., *Soliton in optical metamaterials with parabolic law nonlinearity and spatio-temporal dispersion*, Journal of Optoelectronics and Advanced Materials **16**(11–12), 2014, pp. 1221–1225.
[14] Biswas A., Jawad A.J.M., Zhou Q., *Resonant optical solitons with anti-cubic nonlinearity*, Optik **157**, 2018, pp. 525–531, DOI: 10.1016/j.ijleo.2017.11.125.
[15] Foroutan M., Manafian J., Ranjbaran A., *Solitons in optical metamaterials with anti-cubic law of nonlinearity by ETEM and IGEM*, Journal of the European Optical Society-Rapid Publications **14**, 2018, article 16, DOI: 10.1186/s41476-018-0084-x.
[16] Biswas A., Yasar E., Yildirim Y., Triki H., Zhou Q., Moshokoa S.P., Belic M., *Conservation laws for perturbed solitons in optical metamaterials*, Results in Physics **8**, 2018, pp. 898–902, DOI: 10.1016/j.rinp.2017.12.068.
[17] Biswas A., *Conservations laws for optical solitons with anti-cubic and generalized anti-cubic nonlinearities*, Optik **176**, 2019, pp. 198–201, DOI: 10.1016/j.ijleo.2018.09.074.
[18] Moubissi A.B., Nakkeeran K., Dinda P.T., Kofane T.C., *Non-Lagrangian collective variable approach for optical soliton in fibers*, Journal of Physics A **34**(1), 2001, pp. 129–136, DOI: 10.1088/0305-4470/34/1/310.
[19] Green P., Milovic D., Sarma A.K., Lott D.A., Biswas A., *Dynamics of super-sech solitons in optical fibers*, Journal of Nonlinear Optical Physics & Materials **19**(2), 2010, pp. 339–370, DOI: 10.1142/S0218863510005145.
[20] Biswas A., *Dispersion-managed solitons in optical fibers*, Journal of Optics A **4**(1), 2002, pp. 84–97, DOI: 10.1088/1464-4258/4/1/315.
[21] Hosseini K., Matinfar M., Mirzazadeh M., *Soliton solutions of high-order nonlinear Schrödinger equations with different laws of nonlinearities*, Regular and Chaotic Dynamics **26**(1), 2021, pp. 105–112, DOI: 10.1134/S1560354721010068.
[22] Kudryashov N.A., *Highly dispersive optical solitons of an equation with arbitrary refractive index*, Regular and Chaotic Dynamics **25**(6), 2020, pp. 537–543, DOI: 10.1134/S1560354720060039.
[23] Kudryashov N.A., *Solitary waves of the non-local Schrödinger equation with arbitrary refractive index*, Optik **231**, 2021, article 166443, DOI: 10.1016/j.ijleo.2021.166443.
[24] Kudryashov N.A., Safonova D.V., *Painlevé analysis and traveling wave solutions of the fourth-order differential equation for pulse with non-local nonlinearity*, Optik **227**, 2021, article 166019, DOI: 10.1016/j.ijleo.2020.166019.
[25] Kudryashov N.A., *Almost general solution of the reduced higher-order nonlinear Schrödinger equation*, Optik **230**, 2021, article 166347, DOI: 10.1016/j.ijleo.2021.166347.
[26] Kudryashov N.A., *Solitary waves of the generalized Sasa–Satsuma equation with arbitrary refractive index*, Optik **232**, 2021, article 166540, DOI: 10.1016/j.ijleo.2021.166540.
[27] Kudryashov N.A., *Optical solitons of mathematical model with arbitrary refractive index*, Optik **224**, 2020, article 165391, DOI: 10.1016/j.ijleo.2020.165391.
[28] Kudryashov N.A., *Optical solitons of model with integrable equation for wave packet envelope*, Chaos, Solitons & Fractals **141**, 2020, article 110325, DOI: 10.1016/j.chaos.2020.110325.
[29] Kudryashov N.A., *Optical solitons of the model with arbitrary refractive index*, Optik **224**, 2020, article 165767, DOI: 10.1016/j.ijleo.2020.165767.
[30] Kudryashov N.A., *Highly dispersive optical solitons of equation with various polynomial nonlinearity law*, Chaos, Solitons & Fractals **140**, 2020, article 110202, DOI: 10.1016/j.chaos.2020.110202.

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