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W-shaped and bright optical solitons in negative indexed materials

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ABSTRACT

We investigate a generalized nonlinear Schrödinger equation with higher-order effects such as pseudo-quintic nonlinearity and self-steepening effect. The model applies to the description of ultrashort pulse propagation in nonlinear materials exhibiting a negative index of refraction. Three new types of nonlinearly chirped *W*-shaped soliton solutions are derived for the first time by using the traveling-wave method. The obtained structures have new functional forms that are distinct from the usual W-shaped soliton solution reported within the context of optical fibers. An important characteristic of these envelope solitons is the nonlinear chirp that is directly proportional to the intensity of the pulse. It is shown that these chirped *W*-shaped structures are formed as a result of the exact balance among the group velocity dispersion, the self-steepening effect, and the pseudo-quintic nonlinearity. Exact chirped bright soliton solutions of the model were also obtained under appropriate conditions. Our results may raise the possibility of some experiments and potential applications related to left-handed materials in the presence of self-steepening nonlinearity.

1. Introduction

Ultrashort pulses propagation in optical fibers has been widely studied in recent years due to its numerous applications in telecommunication and ultrafast signal routing systems [1]. The cubic nonlinear Schrö dinger equation (NLSE) is the generic nonlinear equation model that describes the evolution of a picosecond optical pulse in monomode optical fibers, within the slowly varying envelope approximation. An important property of this model is its complete integrability by means of the inverse scattering transform (IST) [2], with the soliton–a localized optical pulse–one the fundamental solutions. Early works mostly concentrated on the media with Kerr nonlinearity, where the refractive index varies linearly with the pulse intensity as [3]: $n = n_0 + n_2 I$, with n_0 being the linear refractive index, n_2 is the coefficient for the nonlinear index, and I is the intensity of the light field. Such uniform Kerr nonlinear media exhibit stable fundamental (single-hump) solitons in one spatial dimension [4–6] and collapse in two and three spatial dimensions [7,8].

Later works focused on the dynamics of femtosecond pulses in optical fibers (≤ 100 fs), for which the standard NLSE becomes not valid. The spectral width of these pulses becomes comparable with the carrier frequency and additional effects should be taken into account [9]. Accordingly, the governing equation should still include higher-order effects such as the third-order dispersion, self-steepening, delayed nonlinear response, quintic nonlinearity, etc. These effects may add new properties to the wave propagation phenomena and also change the physical features and the stability of the NLS soliton. Recently, a kind of NLSE called generalized NLSE is presented and used to describe the propagation of ultrashort pulses in a negative index material [10]. This new wave equation includes correction terms that appear during the propagation of pulses at least a few tens of optical cycles in duration, such as the pseudo-quintic nonlinearity and self-steepening term. Such model has become an important theoretical tool in recent investigations where interesting nonlinear phenomena have been studied in the context of negative index materials [10–13].

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Many novel localized structures, referred to as higher-order (multihump) solitons, have been recently demonstrated experimentally and theoretically in both one- and two-dimensional nonlinear media. Such phenomena include (but are not limited to) dipole solitons (consisting of two peaks) [15–18], multipoles (featuring multiple peaks) [19], *W*-shaped solitons [20], and so on. These nonfundamental soliton structures have the potential to be used as carriers and conduits for data transmission and processing, in the context of all-optical schemes [21].

A particularly interesting type of envelope is the so-called *W* -shaped kind of soliton, which was first reported in an optical fiber medium with higher-order effects [20], and obtained later in a variety of higher-order NLSEs [22,23]. To be specific, this type of pulse shape possesses one hump and two valleys on the hump's two sides. When compared with the fundamental bright and dark solitons, the occurrence of this kind of soliton in nonlinear optics is relatively rare. Nevertheless, all *W*-shaped soliton structures encountered so far analytically in nonlinear fiber media [20–23], were found to be chirp-free and have a functional form expressed in term of the secant hyperbolic function plus a constant background field [see Eq. (33) in Ref. [20]].

In recent years, much attention has been drawn toward envelope soliton pulses with nonlinear chirp [24–29], due to their extensive applications to the design of fiber-optic amplifiers, optical pulse compressors, and solitary-wave-based communications links [30,31]. Usually, the chirped soliton's dynamics is described by the NLS family of equations, which incorporates additional higher-order nonlinear effects to the cubic model.

To the best of our knowledge, the propagation characteristics of nonlinearly chirped *W*-shaped soliton pulses in negative index materials have not been previously studied. In this paper, we present three new types of chirped *W*-shaped soliton solutions, other than the one reported within the context of optical fibers [20], to the generalized NLSE governing the pulse propagation in such systems. We also show that a variety of chirped bright soliton solutions can exists for the equation considered, illustrating the potentially rich set of chirped structures in negative index materials.

The arrangement of the paper is as follows. The generalized nonlinear Schr ödinger equation (GNLSE) for femtosecond pulses propagating in negative index materials will be cited in Section 2. In Section 3, the specific chirp ansatz is assumed and the nonlinear differential equation that governs dynamics of field amplitude is derived by using the traveling-wave method. A rich variety of exact nonlinearly chirped *W*-shaped soliton solutions is presented in Section 4 together with the nonlinear chirp associated with each of these optical pulses. In addition, novel chirped bright soliton solutions will be also given under certain parametric conditions. The paper is concluded in Section 5 by a short summary and discussion.

2. Theoretical model

We study the propagation of ultrashort pulses in negative index materials with higher-order effects such as pseudoquintic nonlinearity and self-steepening effect. The evolution of the pulse envelope is governed by the following GNLSE [10,11]

$$i\psi_\xi + \frac{k}{2}\psi_{\tau\tau} + i\sigma\left(|\psi|^2\psi\right)_\tau + \rho|\psi|^2\psi + 3\delta|\psi|^4\psi = 0, \qquad (1)$$

where $\psi(\xi,\ \tau)$ represents the complex envelope of the electric field, $\tau = ct/\lambda_p$ and $\xi = z/\lambda_p$ are the respective normalized time and propagation distance, where λ_p is the plasma wavelength, and $k,\ \sigma,\ \rho,$ and δ are the real parameters related to group velocity dispersion (GVD), self-steepening, cubic nonlinearity, and pseudo-quintic nonlinearity, respectively. These parameters are defined as [10,11]: $k = \frac{1}{\beta n}\left[\frac{1}{V_g^2} - \alpha\gamma - \beta\frac{\varepsilon\gamma' + \mu\alpha'}{4\pi}\right]$, $\sigma = -\chi^{(3)}\left[\frac{\mu}{2V_g n^2} - \frac{\gamma+\mu}{2n}\right]$, $\rho = \frac{\beta\mu\chi^{(3)}}{2n}$, and $\delta = -\frac{\beta\left(\mu\chi^{(3)}\right)^2}{24n^3}$. Here $V_g = \frac{2n}{\varepsilon\gamma+\mu\alpha}$, $\beta = 2\pi\tilde{\omega} = \frac{2\pi\omega}{\omega_p}$ with ω_p is the plasma frequency, $\alpha = \frac{\partial[\tilde{\omega}\varepsilon(\tilde{\omega})]}{\partial\tilde{\omega}}$, $\alpha' = \frac{\partial^2[\tilde{\omega}\varepsilon(\tilde{\omega})]}{\partial\tilde{\omega}^2}$, $\gamma = \frac{\partial[\tilde{\omega}\mu(\tilde{\omega})]}{\partial\tilde{\omega}}$, and $\gamma' = \frac{\partial^2[\tilde{\omega}\mu(\tilde{\omega})]}{\partial\tilde{\omega}^2}$, where ε denotes dielectric susceptibility, μ is the magnetic permeability, n is the refractive index, $\chi^{(3)}$ is the third order susceptibility of the medium, V_g is the group velocity, and $\tilde{\omega}$ is the normalized frequency.

For the particular case $\sigma = \delta = 0$, the model Eq. (1) reduces to the standard NLSE which has only the terms describing lowest order dispersion and self-phase modulation. If $\delta = 0$, Eq. (1) becomes the derivative NLSE modeling the propagation of NLS soliton in the presence of Kerr dispersion [32].

With higher-order terms, Eq. (1) has been recently studied by many authors. Marklund et al. [12] examined the modulational instability and localization of an ultrashort electromagnetic pulse that is governed by this nonlinear Schrödinger type equation. Zhang and Yi [11] found the exact chirped bright soliton solutions by using a variable parametric method. Additionally, Daoui et al. [13] obtained bright and double-kinked quasi-soliton solutions with nonlinear chirp by employing the traveling-wave method. Moreover, Yang et al. [14] have reported the existence of quasi-soliton solutions for Eq. (1) under the condition $\sigma = 0$. However, the exact nonlinearly chirped *W*-shaped soliton solutions of this model in the presence of all the physical effects have not been reported yet. How to find the exact and new-type solutions having the shape of *W* which are characterized by a nonlinear chirp is an interesting work. Such an attempt appears in what follows.

3. The traveling–wave method and amplitude equation

We are interested to find the exact chirped soliton solutions for Eq. (1). To do so we put the complex field $\psi(\xi,\ \tau)$ in the traveling-wave form [25,28]

$$\psi(\xi,\tau) = u(\zeta)e^{i\Phi(\xi,\tau)}, \qquad (2)$$

with the phase shift $\Phi(\xi,\ \tau)$ as [11]

$$\Phi(\xi,\tau) = E\tau + F\xi + \phi(\zeta) \qquad (3)$$

where $\zeta = \kappa\tau + \lambda\xi$ is the traveling coordinate. Here κ, λ, E, and F are all real parameters. Also $\phi(\zeta)$ denotes a possible nonlinear phase shift which is a real function of ζ. It is worth mentioning that ϕ is a constant parameter for the standard NLSE, while it becomes a nonlinear function of the retarded coordinates ξ and τ in the presence of higher-order terms [11].

Then, the envelope solution above acquires an extra instantaneous frequency shift (i.e., chirp) given by

$$\delta w(\xi,\tau) = -\frac{\partial}{\partial\tau}[E\tau + F\xi + \phi(\zeta)] = -E - \kappa\phi'(\zeta). \qquad (4)$$

where the prime stands for differentiation with respect to ζ. On substituting Eq. (2) along with Eq. (3) in Eq. (1) and separating real and imaginary parts of the resulting equation, one obtains the following coupled equations:

$$-\frac{1}{2}\left(2F + kE^2\right)u - (\lambda + k\kappa E)\phi' u + \frac{1}{2}k\kappa^2 u'' - \frac{1}{2}k\kappa^2\phi'^2 u$$
$$+(\rho - \sigma E)u^3 - \sigma\kappa\phi' u^3 + 3\delta u^5 = 0, \qquad (5)$$

and

$$(\lambda + k\kappa E)u' + \frac{1}{2}k\kappa^2\left(u\phi'' + 2u'\phi'\right) + 3\sigma\kappa u^2 u' = 0. \qquad (6)$$

To solve these equations, we adopt an *ansatz* of the form:

$$\phi' = pu^2 + q, \qquad (7)$$

where q and p are two real parameters that determine the constant and nonlinear chirp parameters, respectively. Inserting the *ansatz*

(7) into Eq. (6), one gets relations of chirp parameters p and q as

$$p=-\frac{3\sigma}{2k\kappa},\ q=-\frac{(\lambda+k\kappa E)}{k\kappa^2}. \tag{8}$$

Accordingly we find that the resultant chirp (4) can be derived as $\delta w(\xi,\tau)=-E-\kappa q-\kappa pu^2$. The later shows that the chirping exhibits a nontrivial form which has an intensity dependent term apart from the linear contribution [with $I=|\psi|^2=u^2$]. The first relation in (8) indicates that the nonlinear chirp is strongly dependent on the self-steepening coefficient σ. In particular, when $\sigma=0$, the localized pulse solutions without chirp are generated in the negative index material.

By means of Eqs. (7) and (8), a nonlinear differential equation for $u(\zeta)$ is readily derived from Eq. (5),

$$u''+\frac{3\left(\sigma^2+8\delta k\right)}{4k^2\kappa^2}u^5+\frac{2[\sigma(\lambda+k\kappa E)+k\kappa(\rho-\sigma E)]}{k^2\kappa^3}u^3+\frac{(\lambda+k\kappa E)^2-k\kappa^2\left(2F+kE^2\right)}{k^2\kappa^4}u=0, \tag{9}$$

This elliptic equation governs the dynamics of field amplitude as it propagates in the negative index material. It is well known that this type of nonlinear differential equations admits many types of exact solutions such as bright soliton, dark soliton, kink, double kink, and bell shaped solutions [28,33,34]. But there exist no known exact analytical soliton solutions with the shape of W to Eq. (9) in the literature. Here, for the first time to our knowledge, we present new types of exact W-shaped soliton solutions to this elliptic equation. These solutions may play an important role in understanding the complicated physical phenomena in negative index materials.

First let us rewrite the preceding equation in a more simplified form. Multiplying Eq. (9) by u_ζ and integrating with respect to ζ, we get

$$\left(u'\right)^2+\frac{\left(\sigma^2+8\delta k\right)}{4k^2\kappa^2}u^6+\frac{\sigma(\lambda+k\kappa E)+k\kappa(\rho-\sigma E)}{k^2\kappa^3}u^4+\frac{(\lambda+k\kappa E)^2-k\kappa^2\left(2F+kE^2\right)}{k^2\kappa^4}u^2+2\mathcal{E}=0, \tag{10}$$

where $\mathcal{E}$ is an arbitrary constant of integration, which coincides with the energy of the anharmonic oscillator [35,36]. Introduction of the change of variable $u^2=v$ in Eq. (10) leads to the following new elliptic equation:

$$v''+\frac{2\left(\sigma^2+8\delta k\right)}{k^2\kappa^2}v^3+\frac{6[\sigma(\lambda+k\kappa E)+k\kappa(\rho-\sigma E)]}{k^2\kappa^3}v^2+\frac{4\left[(\lambda+k\kappa E)^2-k\kappa^2\left(2F+kE^2\right)\right]}{k^2\kappa^4}v+4\mathcal{E}=0, \tag{11}$$

Integrating Eq. (11) for different values of $\mathcal{E}$, we get the intensity function $v(\zeta)$. This waveform when it exists can be substituted in (2) to get the exact chirped solution of the GNLSE given in (1) as

$$\psi(\xi,\tau)=v^{1/2}\exp[i(E\tau+F\xi)+i\phi(\zeta)], \tag{12}$$

where the phase variable ϕ can be obtained from integrating (7) with respect to ζ as

$$\phi=-\frac{(\lambda+k\kappa E)}{k\kappa^2}(\kappa\tau+\lambda\xi)-\frac{3\sigma}{2k\kappa}\int^{\kappa\tau+\lambda\xi}v\,d\zeta+\phi_0. \tag{13}$$

with ϕ_0 being a constant phase.

Based on the preceding results, we also find that the associated chirp can be expressed in term of v as

$$\delta w(\xi,\tau)=-E+\frac{(\lambda+k\kappa E)}{k\kappa}+\frac{3\sigma}{2k}v. \tag{14}$$

Eq. (12) together with the relations (13) and (14) are the central theoretical results representing the general form of exact nonlinearly chirped solutions for the GNLSE (1) and their associated nonlinear chirp. If one can determine the functions $v(\zeta)$ from the nonlinear differential equation (11), then we can construct the chirped solutions of the model considered and find their corresponding chirping based on the general wave form (12) and expression (14). In the following, based on solving the nonlinear differential equation above, we mainly discuss novel chirped localized solutions for Eq. (1) that are firstly reported in this paper.

4. Novel chirped *W*-shaped soliton solutions

In this section, we present three new types of exact solutions of the GNLSE (1) that describe nonlinearly chirped W-shaped soliton propagation with a pronounced platform underneath it.

4.1. *First chirped* W*-shaped soliton solution*

The first solution we have found for Eq. (11) reads

$$v(\zeta)=A\left[1-\frac{3}{2}\operatorname{sech}^2(\mu\zeta)\right], \tag{15}$$

which is obtained when choosing a zero value of energy ($\mathcal{E}=0$). The amplitude and width of the pulse are defined by

$$A=-\frac{2\left[(\lambda+k\kappa E)^2-k\kappa^2\left(2F+kE^2\right)\right]}{3\kappa[\sigma(\lambda+k\kappa E)+k\kappa(\rho-\sigma E)]}, \tag{16}$$

$$\mu^2=\frac{(\lambda+k\kappa E)^2-k\kappa^2\left(2F+kE^2\right)}{k^2\kappa^4}, \tag{17}$$

for the relevant constraint condition:

$$\delta=-\frac{\sigma^2}{8k}. \tag{18}$$

Making use of all these findings, we can present the chirped soliton solution of Eq. (1) as

$$\psi(\xi,\tau)=\left[-\frac{2\left[(\lambda+k\kappa E)^2-k\kappa^2\left(2F+kE^2\right)\right]}{3\kappa[\sigma(\lambda+k\kappa E)+k\kappa(\rho-\sigma E)]}\left\{1-\frac{3}{2}\operatorname{sech}^2\left[\sqrt{\frac{(\lambda+k\kappa E)^2-k\kappa^2\left(2F+kE^2\right)}{k^2\kappa^4}}(\kappa\tau+\lambda\xi)\right]\right\}\right]^{1/2}\times\exp[i(E\tau+F\xi+\phi(\zeta))], \tag{19}$$

provided that $(\lambda+k\kappa E)^2-k\kappa^2\left(2F+kE^2\right)>0$ and $\kappa[\sigma(\lambda+k\kappa E)+k\kappa(\rho-\sigma E)]<0$ in order to ensure the pulse width and amplitude to be real.

The associated chirp can be obtained readily as

$$\delta w(\xi,\tau)=-E+\frac{(\lambda+k\kappa E)}{k\kappa}-\frac{\sigma\left[(\lambda+k\kappa E)^2-k\kappa^2\left(2F+kE^2\right)\right]}{k\kappa[\sigma(\lambda+k\kappa E)+k\kappa(\rho-\sigma E)]}\left[1-\frac{3}{2}\operatorname{sech}^2(\mu\zeta)\right]. \tag{20}$$

As it can be seen, the existence of this nonlinearly chirped solution requires the condition (18) to be satisfied, which shows a subtle balance among the GVD term, pseudo-quintic nonlinearity, and self-steepening effect. Physically, this requirement implies that the present solution (19) can exist in abnormal (normal) dispersion for self-focusing (defocusing) nonlinearity. This is different from that in ordinary materials. It is worthy to mention that in a negative index material, the sign of GVD can be positive or negative and self-steepening characterizes the front of the pulse, different from the case of ordinary materials [10].

Fig. 1(a) shows the propagation of the chirped solution (19) in the presence of higher-order effects. Here we have taken the following values for the model parameters [14]: $k = -0.7954$, $\rho = 1.2566 \times 10^{-10}$, and $\delta = 0.6983 \times 10^{-21}$. To satisfy the parametric condition (18), we set $\sigma = 0.6666 \times 10^{-10}$. The other parameters are chosen as $\kappa = 1$, $\lambda = -\frac{1}{8}$, $E = -2.0423$ and $F = \frac{1}{2}$. From this figure, one can clearly see that the chirped soliton pulse takes the shape of W with a pronounced platform underneath. The chirping profile for this W-shaped structure is shown in Fig. 1(b) (for $\xi = 0$). One can see that this chirp is localized in time and has a maximum at the center of the pulse.

On the contrary, if $(\lambda + k\kappa E)^2 - k\kappa^2(2F + kE^2) < 0$ and $\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)] > 0$, one can obtain an exact analytical solution to Eq. (1) representing nonlinearly chirped bright solitons in the form

$$\psi(\xi,\tau) = \sqrt{\frac{k\kappa^2(2F + kE^2) - (\lambda + k\kappa E)^2}{\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)]}} \operatorname{sech}\left[\sqrt{\frac{k\kappa^2(2F + kE^2) - (\lambda + k\kappa E)^2}{k^2\kappa^4}}(\kappa\tau + \lambda\xi)\right] \times \exp[i(E\tau + F\xi + \phi(\zeta))], \tag{21}$$

The corresponding chirping is given by

$$\delta w(\xi,\tau) = -E + \frac{(\lambda + k\kappa E)}{k\kappa} - \frac{3\sigma\left[k\kappa^2(2F + kE^2) - (\lambda + k\kappa E)^2\right]}{2k\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)]} \operatorname{sech}^2\left[\sqrt{\frac{k\kappa^2(2F + kE^2) - (\lambda + k\kappa E)^2}{k^2\kappa^4}}(\kappa\tau + \lambda\xi)\right]. \tag{22}$$

As it can be seen, the chirped bright soliton above has the same sech-profile as the one for the cubic NLSE, except the chirping will show a nonlinear behavior.

For the sake of completeness we now discuss an important particular case when $\lambda + k\kappa E = 0$ and $2F + kE^2 = k\kappa^2$. Taking this limit, we can further reduce the envelope function solution (21) to the waveform

$$\psi(\xi,\tau) = \kappa\sqrt{\frac{k}{\rho - \sigma E}}\, sech(\kappa\tau + \lambda\xi)e^{i[E\tau + F\xi + \phi(\zeta)]}, \tag{23}$$

with $k(\rho - \sigma E) > 0$. In this solution, the phase modification $\phi(\zeta)$ can be determined by using (13) and (23) as

$$\phi = -\frac{3\sigma\kappa}{2(\rho - \sigma E)}\tanh(\kappa\tau + \lambda\xi) + \phi_0, \tag{24}$$

while the associated chirp shown in Eq. (22) can be significantly simplified to

$$\delta w(\xi,\tau) = -E - \frac{3\sigma\kappa^2}{2(\rho - \sigma E)}\operatorname{sech}^2(\kappa\tau + \lambda\xi). \tag{25}$$

The chirped bright solution (23) with (24) and its associated chirp (25) is exactly the form of the solution derived by Zhang and Yi [11] under the same parametric condition (18) by adopting another methodology [see Eqs. (18) and (19) in Ref. [11]]. We can conclude that our bright solution (21) and its associated chirp (22) generalize the earlier reported results [11] to the case of $\lambda + k\kappa E \neq 0$ and $2F + kE^2 \neq k\kappa^2$. Because of the appearance of several parameters related to the amplitude and width of the solution (21), this provides a possible way to tune experimentally the chirped soliton pulse by choosing different values of them.

4.2. Second chirped W-shaped soliton solution

We now present a different interesting exact soliton solution for Eq. (11). This solution is of the form

$$v(\zeta) = \lambda\left[1 - \frac{3\operatorname{sech}(\eta\zeta)}{1 + \operatorname{sech}(\eta\zeta)}\right], \tag{26}$$

where

$$\lambda = -\frac{2\left[(\lambda + k\kappa E)^2 - k\kappa^2(2F + kE^2)\right]}{3\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)]}, \tag{27}$$

$$\eta^2 = \frac{4\left[(\lambda + k\kappa E)^2 - k\kappa^2(2F + kE^2)\right]}{k^2\kappa^4}. \tag{28}$$

under the same constraint stated in (18). Here, the value of energy $\mathcal{E}$ is also chosen equal to zero. Combining Eqs. (12) and (26) with (27) and (28), we find that the chirped solution of Eq. (1) can be written as

$$\psi(\xi,\tau) = \left[-\frac{2\left[(\lambda + k\kappa E)^2 - k\kappa^2(2F + kE^2)\right]}{3\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)]}\left\{1 - \frac{3\operatorname{sech}[\eta(\kappa\tau + \lambda\xi)]}{1 + \operatorname{sech}[\eta(\kappa\tau + \lambda\xi)]}\right\}\right]^{1/2} e^{i[E\tau + F\xi + \phi(\zeta)]}, \tag{29}$$

provided that $(\lambda + k\kappa E)^2 - k\kappa^2(2F + kE^2) > 0$ and $\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)] < 0$. Correspondingly, the chirping reads

$$\delta w(\xi,\tau) = -E + \frac{(\lambda + k\kappa E)}{k\kappa} - \frac{\sigma\left[(\lambda + k\kappa E)^2 - k\kappa^2(2F + kE^2)\right]}{k\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)]}\left[1 - \frac{3\operatorname{sech}(\eta\zeta)}{1 + \operatorname{sech}(\eta\zeta)}\right]. \tag{30}$$

Fig. 2 (a) depicts the evolution behavior of the chirped solution (29) for the same values of parameters as in Fig. 1. From it, one can clearly see that this solution presents a W-shaped soliton wave. The corresponding chirping for this structure is shown in Fig. 2 (b) (for $\xi = 0$). One can see that the chirp has a maximum at the center of the pulse and it saturates at the same finite value as $\tau \to \pm\infty$.

In the opposite limit, when $(\lambda + k\kappa E)^2 - k\kappa^2(2F + kE^2) < 0$ and $\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)] > 0$, one can obtain an exact chirped bright soliton solution to Eq. (1) of the form

$$\psi(\xi,\tau) = \left[\frac{2\left[k\kappa^2(2F + kE^2) - (\lambda + k\kappa E)^2\right]}{\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)]}\frac{\operatorname{sech}[\eta(\kappa\tau + \lambda\xi)]}{1 + \operatorname{sech}[\eta(\kappa\tau + \lambda\xi)]}\right]^{1/2} e^{i[E\tau + F\xi + \phi(\zeta)]}, \tag{31}$$

with the parameter η as follows:

$$\eta^2 = \frac{4\left[k\kappa^2(2F + kE^2) - (\lambda + k\kappa E)^2\right]}{k^2\kappa^4}. \tag{32}$$

For this case, the associated chirp takes the form

$$\delta w(\xi,\tau) = -E + \frac{(\lambda + k\kappa E)}{k\kappa} - \frac{3\sigma\left[(\lambda + k\kappa E)^2 - k\kappa^2(2F + kE^2)\right]}{k\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)]}\frac{\operatorname{sech}(\eta\zeta)}{1 + \operatorname{sech}(\eta\zeta)}. \tag{33}$$

4.3. Third chirped W-shaped soliton solution

We find that Eq. (11) possesses another exact soliton solution of the form

$$v(\zeta) = B\left[1 - \frac{12\operatorname{sech}^2(s\zeta)}{4 - [1 - \tanh(s\zeta)]^2} + \frac{24\, sech^4(s\zeta)}{\left(4 - [1 - \tanh(s\zeta)]^2\right)^2}\right], \tag{34}$$

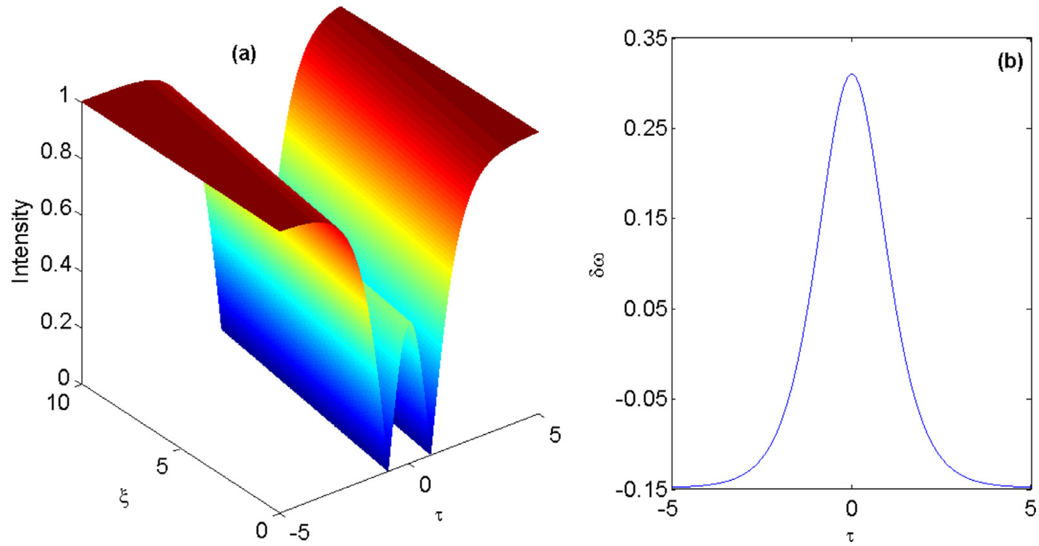

Fig. 1. (a) Evolution of intensity wave profile of the *W*-shaped soliton pulse as computed from (19) of Eq. (1) and (b) profile of chirping given by Eq. (20). Here we have used the parameters values $k = -0.7954$, $\rho = 1.2566 \times 10^{-10}$, $\delta = 0.6983 \times 10^{-21}$, $\sigma = 0.6666 \times 10^{-10}$, $\kappa = 1$, $\lambda = -\frac{1}{8}$, $E = -2.0423$ and $F = \frac{1}{2}$. The soliton intensity is normalized by $|\psi(\xi, \tau)|^2/A$.

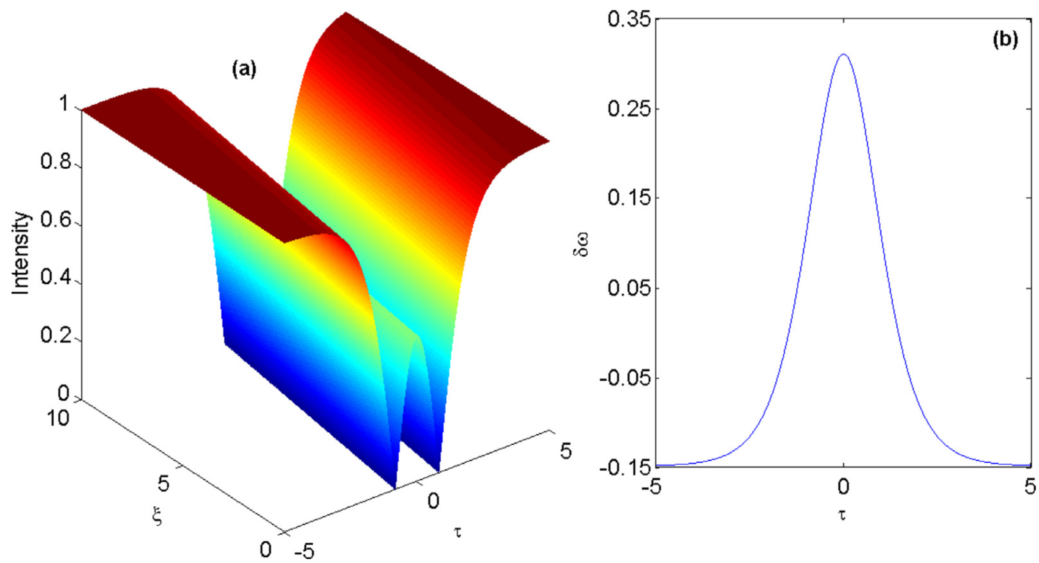

Fig. 2. (a) Evolution of intensity wave profile of the *W*-shaped soliton pulse as computed from (29) of Eq. (1) and (b) profile of chirping given by Eq. (30). Here we have used the same values of parameters as in Fig. 1. The soliton intensity is normalized by $|\psi(\xi, \tau)|^2/\lambda$.

where

$$B = -\frac{2\left[(\lambda + k\kappa E)^2 - k\kappa^2\left(2F + kE^2\right)\right]}{3\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)]}, \tag{35}$$

$$s^2 = \frac{(\lambda + k\kappa E)^2 - k\kappa^2\left(2F + kE^2\right)}{k^2\kappa^4}, \tag{36}$$

under the special condition (18). Here, we also choose the energy $\mathcal{E}$ as a zero value. Thus, the complete chirped soliton solution to Eq. (1) can be written as

$$\psi(\xi, \tau) = \left[B\left\{1 - \frac{12\,\mathrm{sech}^2[s(\kappa\tau + \lambda\xi)]}{4 - [1 - \tanh[s(\kappa\tau + \lambda\xi)]]^2} + \frac{24\,\mathrm{sech}^4[s(\kappa\tau+\lambda\xi)]}{\left(4 - [1 - \tanh[s(\kappa\tau+\lambda\xi)]]^2\right)^2}\right\}\right]^{1/2} e^{i[E\tau + F\xi + \phi(\zeta)]}. \tag{37}$$

provided that $(\lambda + k\kappa E)^2 - k\kappa^2\left(2F + kE^2\right) > 0$ and $\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)] < 0$ to ensure the width s and amplitude B of the pulse to be real.

The corresponding chirping will be of the form

$$\delta w(\xi, \tau) = -E + \frac{(\lambda + k\kappa E)}{k\kappa} + \frac{3\sigma B}{2k}\left[1 - \frac{12\,\mathrm{sech}^2(s\zeta)}{4 - [1 - \tanh(s\zeta)]^2} + \frac{24\,\mathrm{sech}^4(s\zeta)}{\left(4 - [1 - \tanh(s\zeta)]^2\right)^2}\right]. \tag{38}$$

Fig. 3(a) illustrates the nonlinear evolution behavior of the chirped solution (37) for the same values of parameters as in Fig. 1. As one can see, this solution represents a *W*-shaped pulse that propagates in the metamaterial in the presence of higher-order effects. The chirp associated with this nonlinear structure is shown in Fig. 3(b) (for $\xi = 0$), which has a minimum at the center of the pulse and saturates at the same finite value as $\tau \to \pm\infty$.

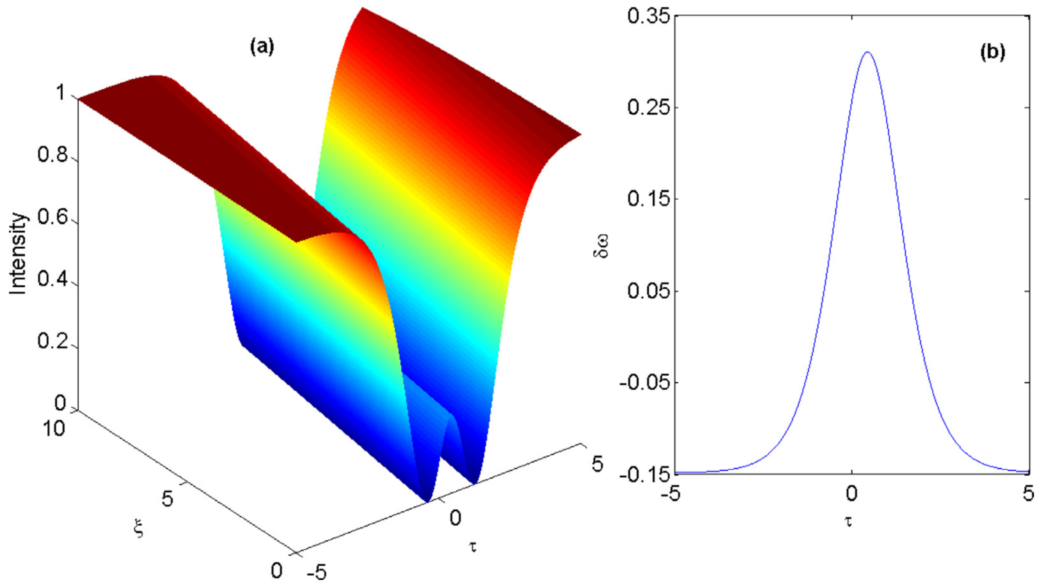

Fig. 3. (a) Evolution of intensity wave profile of the *W*-shaped soliton pulse as computed from (37) of Eq. (1) and (b) profile of chirping given by Eq. (38). Here we have used the same values of parameters as in Figs. 1 and 2. The soliton intensity is normalized by $|\psi(\xi,\tau)|^2/B$.

On the contrary, when $(\lambda + k\kappa E)^2 - k\kappa^2(2F + kE^2) < 0$ and $\kappa[\sigma(\lambda + k\kappa E) + k\kappa(\rho - \sigma E)] > 0$, we can obtain an exact chirped bright soliton solution to Eq. (1) of the form

$$\psi(\xi,\tau) = \left[\frac{8\left[k\kappa^2(2F+kE^2) - (\lambda+k\kappa E)^2\right]}{\kappa[\sigma(\lambda+k\kappa E)+k\kappa(\rho-\sigma E)]} \left\{ \frac{\operatorname{sech}^2[s(\kappa\tau+\lambda\xi)]}{4-[1-\tanh[s(\kappa\tau+\lambda\xi)]]^2} - \frac{2\operatorname{sech}^4[s(\kappa\tau+\lambda\xi)]}{\left(4-[1-\tanh[s(\kappa\tau+\lambda\xi)]]^2\right)^2} \right\} \right]^{1/2} \times \exp[i(E\tau+F\xi+\phi(\zeta))], \quad (39)$$

with the following parameter:

$$s^2 = \frac{k\kappa^2(2F+kE^2) - (\lambda+k\kappa E)^2}{k^2\kappa^4}.$$

For this case, the chirping can be written as

$$\delta w(\xi,\tau) = -E + \frac{(\lambda+k\kappa E)}{k\kappa} + \frac{12\sigma\left[k\kappa^2(2F+kE^2)-(\lambda+k\kappa E)^2\right]}{k\kappa[\sigma(\lambda+k\kappa E)+k\kappa(\rho-\sigma E)]} \times \left\{ \frac{\operatorname{sech}^2[s(\kappa\tau+\lambda\xi)]}{4-[1-\tanh[s(\kappa\tau+\lambda\xi)]]^2} - \frac{2\operatorname{sech}^4[s(\kappa\tau+\lambda\xi)]}{\left(4-[1-\tanh[s(\kappa\tau+\lambda\xi)]]^2\right)^2} \right\}. \quad (40)$$

Remarkably, the nonlinearly chirped structures above exhibit specific functional forms, which differ from the usual *W*-shaped solution reported in Ref. [20]. Stability of these privileged exact chirped solutions against small perturbations, such as amplitude perturbation, random noises and the slight violation of the parametric conditions, is a crucial issue as it is linked with the experimental observation of them. It is relevant to mention that only stable (or weakly unstable) solitary waves can be observed experimentally [37]. This analysis can be done by the linear stability theory and numerical simulations of the solutions with perturbations initially implanted. However, a detailed analysis of the stability of these chirped solutions is beyond the scope of this work.

5. Conclusions

To conclude, for the first time, we derived three new types of nonlinearly chirped *W*-shaped soliton solutions in negative index materials exhibiting higher-order effects such as pseudo-quintic nonlinearity and self-steepening effect. These chirped nonlinear structures have been derived under a specific condition between the material parameters by using the traveling-wave method. Interestingly, the newly found chirped soliton pulses possess a functional form which is different from the one reported in the setting of optical fibers. It is shown that the chirp associated with each of these soliton pulses has a nontrivial form which includes an intensity dependent chirping term apart from the linear contribution. The various nonlinearly chirped bright soliton solutions have also been determined for the model considered. It is shown that some previously known chirped bright soliton solutions of the basic equation can be recovered in a particular limit. The obtained results constitute the first analytical demonstration of propagation of nonlinearly chirped *W*-shaped soliton pulses in a negative index material. These chirped solutions may further raise the possibility of some experiments and potential applications related to left-handed materials in the presence of pseudo-quintic nonlinearity and self-steepening effect.

For instance, these solutions which can represent bits of information can be of particular interest in the investigation of the transmission properties of optical pulses in left-handed media. It is worth to point out that, recently a mechanism of the soliton generation in nonlinear active metamaterials by means of metastructure consisting of a ring resonator formed by a microwave amplifier loaded with a left-handed transmission line was demonstrated experimentally [38]. In this setting, a variety of nonlinear effects including the generation of envelope solitons was observed. Very recently, Shen et al. have investigated the properties of the nonlinear wave forms that arise in a nonlinear left-handed transmission line [39]. They have found that bright and dark solitons can propagate undistorted over a long propagation distance in such left-handed transmission lines, in the case of effectively self-focusing and self-defocusing nonlinearities, respectively. The obtained solutions will also be useful for the study of soliton interactions in left-handed media under the influence of perturbations. Moreover, chirped pulses possess extensive applications in pulse compression or amplification, and thus they are particularly useful in the design of fiber optic amplifiers, optical pulse compressors and solitary wave-based communications links [28,30,31]. We would like to mention that in the present system, the self-steepening effect is essential for the soliton wave packets to exhibit a nonlinear chirp, while the role of the fifth-order nonlinearity can be essential for the physical features and the stability of optical soliton propagation.

It would be particularly relevant to extend the analysis developed here to more general situations, which concern the propagation of W-shaped solitons in an inhomogeneous negative index material. In this case, the variable-coefficient GNLSE gives a suitable description of soliton dynamics, in which the physical parameters are dependent on the propagation distance. Different from the nonlinear evolution equations with constant coefficients, the study of *W*-shaped solitons in variable-coefficient ones is more complicated and can show some novel features. It is relevant to mention that, in realistic systems, no media is homogeneous due to the existence of some nonuniformities which often lead to inhomogeneous effects such variable dispersion and nonlinearity. Studies of the existence and stability properties of nonlinearly chirped W-shaped solitons in negative index materials within the context of the present model with distributed coefficients will be deferred to future work.

Conflict of interest

The authors also declare that there is no conflict of interest.

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